

CLUSTER ALGEBRAS OF FINITE TYPE VIA COXETER ELEMENTS AND PRINCIPAL MINORS

SHIH-WEI YANG AND ANDREI ZELEVINSKY

To Bertram Kostant on the occasion of his 80th birthday

ABSTRACT. We give a uniform geometric realization for the cluster algebra of an arbitrary finite type with principal coefficients at an arbitrary acyclic seed. This algebra is realized as the coordinate ring of a certain reduced double Bruhat cell in the simply connected semisimple algebraic group of the same Cartan-Killing type. In this realization, the cluster variables appear as certain (generalized) principal minors.

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1. INTRODUCTION AND MAIN RESULTS

The theory of cluster algebras has a lot in common with the theory of Kac-Moody algebras. In both instances, the structure of an algebra in question is encoded by a square integer matrix: a generalized Cartan matrix A in the Kac-Moody case, and an exchange matrix B in the case of cluster algebras. (Note an important distinction between the two cases: the sign pattern of matrix entries is symmetric for A but skew-symmetric for B .) In both instances, there is a natural notion of finite type, and the algebras of finite type share the same Cartan-Killing classification. This occasionally leads to an intriguing collision of two types of symmetry: the coordinate ring of a variety associated with a semisimple algebraic group G may carry a natural cluster algebra structure whose Cartan-Killing type is completely different from that of G . For example, the base affine space G/N for $G = SL_5$ inherits the symmetry

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of type A_4 from G , but the coordinate ring $\mathbb{C}[G/N]$ has a natural cluster algebra structure of type D_6 , see [2, Proposition 2.26].

One way to relate the two classifications to each other is by exhibiting a uniform construction of a variety associated with a semisimple group G whose coordinate ring naturally carries a cluster algebra structure of the same Cartan-Killing type. Such a construction was given in [2, Example 2.24] with the variety in question being the double Bruhat cell $G^{c,c^{-1}}$, where c is a Coxeter element in the Weyl group of G (here and in the sequel, we assume G to be simply connected). This result serves as a point of departure for the current paper. Here we improve on it in the following two aspects.

First, there are many non-isomorphic cluster algebras of the same type differing from each other by the choice of a coefficient system. From this perspective, there is nothing especially distinguished about the coefficient system for $\mathbb{C}[G^{c,c^{-1}}]$. On the other hand, as shown in [10], with any exchange matrix one can associate an important system of *principal* coefficients which in a certain sense controls all the other possible choices of coefficients. In this paper we show that one can realize the cluster algebra of finite type with principal coefficients at an arbitrary acyclic initial cluster by replacing the double cell $G^{c,c^{-1}}$ with its *reduced* version $L^{c,c^{-1}}$ introduced in [3].

Second and perhaps more importantly, the treatment in [2] was limited to the description of the initial cluster in $\mathbb{C}[G^{c,c^{-1}}]$, with no information provided about the rest of the cluster variables. Here we compute explicitly all cluster variables in $\mathbb{C}[L^{c,c^{-1}}]$. They turn out to be an interesting special family of principal generalized minors.

Before stating the general results, we present a motivating example. Let $G = SL_{n+1}(\mathbb{C})$ be of type A_n , and let $c = s_1 \cdots s_n$, in the standard numbering of simple roots. In this case $L^{c,c^{-1}}$ is the subvariety of $SL_{n+1}(\mathbb{C})$ consisting of tridiagonal matrices of the form

$$(1.1) \quad M = \begin{pmatrix} v_1 & y_1 & 0 & \cdots & 0 \\ 1 & v_2 & y_2 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & y_n \\ 0 & \cdots & 0 & 1 & v_{n+1} \end{pmatrix}$$

with all y_1, \dots, y_n non-zero. Let $x_{[i,j]} \in \mathbb{C}[L^{c,c^{-1}}]$ denote the regular function on $L^{c,c^{-1}}$ given by the minor with rows and columns $i, i+1, \dots, j$, with the convention that $x_{[i,j]} = 1$ unless $1 \leq i \leq j \leq n+1$. For instance, we have $x_{[i,i]} = v_i$, and $x_{[1,n+1]} = \det(M) = 1$.

In the following result we use the terminology on cluster algebras introduced in [10]; the relevant notions will be recalled in Section 4.

Theorem 1.1.

- (1) *The algebra $\mathcal{A} = \mathbb{C}[L^{c,c^{-1}}]$ is the cluster algebra of type A_n with principal coefficients at the initial seed $(\mathbf{x}, \mathbf{y}, B)$ given by*

$$\mathbf{x} = (x_{[1,1]}, x_{[1,2]}, \dots, x_{[1,n]}),$$

$$\mathbf{y} = (y_1, \dots, y_n),$$

$$B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 & 0 \end{pmatrix}.$$

(2) The set of cluster variables in \mathcal{A} is $\{x_{[i,j]} : 1 \leq i \leq j \leq n+1, (i,j) \neq (1, n+1)\}$.

(3) The exchange relations in \mathcal{A} are:

$$(1.2) \quad x_{[i,k]}x_{[j,\ell]} = y_{j-1}y_j \cdots y_k x_{[i,j-2]}x_{[k+2,\ell]} + x_{[i,\ell]}x_{[j,k]}$$

for $1 \leq i \leq j-1 \leq k \leq \ell-1 \leq n$.

Note that among the relations (1.2), there are the exchange relations from the initial cluster, which can be rewritten as follows:

$$(1.3) \quad x_{[1,k+1]} = v_{k+1}x_{[1,k]} - y_k x_{[1,k-1]} \quad (k = 1, \dots, n).$$

These relations play a fundamental part in the classical theory of orthogonal polynomials in one variable. Thus, the cluster algebra in Theorem 1.1 can be viewed as some kind of “enveloping algebra” for this theory. Note also that the variety $L^{c,c^{-1}}$ is very close to the variety of tridiagonal matrices used by B. Kostant [13] in his study of a generalized Toda lattice.

Our main result is a generalization of Theorem 1.1 to the case of an arbitrary simply connected semisimple complex group G and an arbitrary Coxeter element c in its Weyl group W . We use the terminology and notation in [3, Sections 4.2, 4.3] (more details are given in Sections 2 and 3). Recall that W is a finite Coxeter group generated by the simple reflections s_i ($i \in I$), and that $c = s_{i_1} \cdots s_{i_n}$ for some permutation (i_1, \dots, i_n) of the index set I . For $i, j \in I$, we will write $i \prec_c j$ if i and j are joined by an edge in the Coxeter graph (that is, if the Cartan matrix entry $a_{i,j}$ is non-zero), and s_i precedes s_j in the factorization of c . We associate to c a skew-symmetrizable matrix $B(c) = (b_{i,j})_{i,j \in I}$ by setting

$$(1.4) \quad b_{i,j} = \begin{cases} -a_{i,j} & \text{if } i \prec_c j; \\ a_{i,j} & \text{if } j \prec_c i; \\ 0 & \text{otherwise.} \end{cases}$$

Recall from [5, 3] the finite family of (generalized) minors $\Delta_{\gamma,\delta} \in \mathbb{C}[G]$ labeled by pairs of weights γ, δ belonging to the W -orbit of the same fundamental weight ω_i ($i \in I$). We call a minor $\Delta_{\gamma,\delta}$ *principal* if $\gamma = \delta$; note that this terminology differs from that in [5], where “principal” referred only to the minors $\Delta_{\omega_i,\omega_i}$. We will use the following notation:

$$(1.5) \quad x_{\gamma;c} \in \mathbb{C}[L^{c,c^{-1}}] \text{ is the restriction of } \Delta_{\gamma,\gamma} \text{ to the reduced double cell } L^{c,c^{-1}}.$$

As shown in [5, 3], each of the minors $\Delta_{\omega_i, c\omega_i}$ vanishes nowhere on $L^{c, c^{-1}}$, so can be viewed as an invertible element of $\mathbb{C}[L^{c, c^{-1}}]$. For $j \in I$, we use the following notation:

$$(1.6) \quad y_{j;c} \in \mathbb{C}[L^{c, c^{-1}}] \text{ is the restriction to } L^{c, c^{-1}} \text{ of the product } \Delta_{\omega_j, c\omega_j} \prod_{i \prec_c j} \Delta_{\omega_i, c\omega_i}^{a_{i,j}}.$$

Now we are ready to generalize Part (1) of Theorem 1.1.

Theorem 1.2. *The coordinate ring $\mathcal{A}(c) = \mathbb{C}[L^{c, c^{-1}}]$ is the cluster algebra with principal coefficients at the initial seed $(\mathbf{x}, \mathbf{y}, B)$ given by*

$$\mathbf{x} = (x_{\omega_i; c} : i \in I), \quad \mathbf{y} = (y_{i; c} : i \in I), \quad B = B(c).$$

Furthermore, $\mathcal{A}(c)$ is of finite type, and its Cartan-Killing type is the same as that of G .

It is easy to see that every acyclic exchange matrix of finite type is of the form $B(c)$ for some Coxeter element c . Thus Theorem 1.2 provides a realization of any cluster algebra of finite type with principal coefficients at an arbitrary acyclic seed.

We next generalize Part (2) of Theorem 1.1, by showing that the set of all cluster variables in $\mathcal{A}(c)$ is of the form $\{x_{\gamma; c} : \gamma \in \Pi(c)\}$ for some set of weights $\Pi(c)$. To describe $\Pi(c)$, we need to develop some combinatorics related to the action on weights by the cyclic group $\langle c \rangle$ generated by c . Interestingly, this combinatorics turns out to be very close to the one developed in the classical paper [12] by B. Kostant, and used recently in [11] for (seemingly) completely different purposes.

We consider the usual partial order on weights: $\gamma \geq \delta$ if $\gamma - \delta$ is a nonnegative integer linear combination of simple roots. We also denote by $i \mapsto i^*$ the involution on I given by

$$(1.7) \quad \omega_{i^*} = -w_o(\omega_i),$$

where, as usual, w_o is the longest element of W .

Proposition 1.3. *For every $i \in I$, there is a positive integer $h(i; c)$ such that*

$$(1.8) \quad \omega_i > c\omega_i > c^2\omega_i > \cdots > c^{h(i; c)}\omega_i = -\omega_{i^*}.$$

Theorem 1.4. *The set of cluster variables in $\mathcal{A}(c)$ is $\{x_{\gamma; c} : \gamma \in \Pi(c)\}$, where the set of weights $\Pi(c)$ is given by*

$$(1.9) \quad \Pi(c) = \{c^m \omega_i : i \in I, 0 \leq m \leq h(i; c)\}.$$

As for Part (3) of Theorem 1.1, in the general case we give explicit formulas not for all exchange relations but only for the *primitive* ones, that is, those in which one of the products of cluster variables on the right-hand side is equal to 1.

Theorem 1.5. *The primitive exchange relations in $\mathcal{A}(c)$ are exactly the following:*

$$(1.10) \quad x_{-\omega_k; c} x_{\omega_k; c} = y_{k; c} \prod_{i \prec_c k} x_{\omega_i; c}^{-a_{i,k}} \prod_{k \prec_c i} x_{-\omega_i; c}^{-a_{i,k}} + 1;$$

$$(1.11) \quad x_{c^{m-1}\omega_k; c} x_{c^m\omega_k; c} = \prod_{i \prec_c k} x_{c^{m-1}\omega_i; c}^{-a_{i,k}} \prod_{k \prec_c i} x_{c^{m-1}\omega_i; c}^{-a_{i,k}} + \prod_{j \in I} y_{j; c}^{[c^{m-1}\omega_k - c^m\omega_k : \alpha_j]},$$

where $k \in I$, $1 \leq m \leq h(k; c)$, and $[\gamma : \alpha_j]$ stands for the coefficient of α_j in the expansion of a weight γ in the basis of simple roots.

The fact that the weights $c^{m-1}\omega_i$ and $c^m\omega_i$ appearing in the right hand side of (1.11) belong to $\Pi(c)$, is guaranteed by the following key property of the numbers $h(i; c)$.

Proposition 1.6. *If $i \prec_c j$ then we have*

$$(1.12) \quad h(i; c) - h(j; c) = \begin{cases} 1 & \text{if } j^\star \prec_c i^\star; \\ 0 & \text{if } i^\star \prec_c j^\star. \end{cases}$$

Note that the numbers $h(i; c)$ are uniquely determined by the relations (1.12) combined with the following property.

Proposition 1.7. *For every $i \in I$, the sum $h(i; c) + h(i^\star; c)$ is equal to the Coxeter number of the connected component of I containing i and i^\star .*

In what follows, we identify the index set I with $[1, n] = \{1, \dots, n\}$ so that the (fixed) Coxeter element c has the form $c = s_1 \cdots s_n$. Recall from [10, (7.2), (7.3)] that associated to every cluster variable z is the *denominator vector* $\mathbf{d}_{z; \mathbf{x}} = (d_1, \dots, d_n) \in \mathbb{Z}^n$ with respect to a cluster \mathbf{x} . Namely, z can be uniquely written in the form

$$(1.13) \quad z = \frac{N(x_1, \dots, x_n)}{x_1^{d_1} \cdots x_n^{d_n}},$$

where $N(x_1, \dots, x_n)$ is a polynomial not divisible by any cluster variable $x_i \in \mathbf{x}$. The following result gives an explicit formula for the denominator vector of any cluster variable $z \in \mathcal{A}(c)$ with respect to the initial cluster.

Theorem 1.8. *Let \mathbf{x} be the initial cluster in Theorem 1.2, and let $z = x_{\gamma; c}$ be a cluster variable not belonging to \mathbf{x} . Identifying \mathbb{Z}^n with the root lattice by means of the basis $\alpha_1, \dots, \alpha_n$ of simple roots, the denominator vector $\mathbf{d}_{z; \mathbf{x}}$ gets identified with $c^{-1}\gamma - \gamma$.*

The following corollary agrees with [10, Conjecture 7.4 (i), (ii)].

Corollary 1.9. *In the situation of Theorem 1.8, all the components d_i of the denominator vector $\mathbf{d}_{z; \mathbf{x}}$ are nonnegative. Furthermore, $d_i = 0$ if and only if z and the initial cluster variable $x_{\omega_i; c}$ belong to the same cluster in $\mathcal{A}(c)$.*

According to [10, Corollary 6.3] (see also Section 4 below), for any choice of coefficients, expressing any cluster variable z in terms of an initial cluster \mathbf{x} only requires knowing the **g**-vector $\mathbf{g}_{z; \mathbf{x}} \in \mathbb{Z}^n$ and the *F*-polynomial $F_{z; \mathbf{x}} \in \mathbb{Z}[t_1, \dots, t_n]$. Our realization of cluster variables allows us to obtain explicit expressions for their **g**-vectors and *F*-polynomials. We start with **g**-vectors.

Theorem 1.10. *Let \mathbf{x} be the initial cluster in Theorem 1.2, and let $z = x_{\gamma; c}$ for some $\gamma \in \Pi(c)$. Identifying \mathbb{Z}^n with the weight lattice by means of the basis $\omega_1, \dots, \omega_n$ of fundamental weights, the **g**-vector $\mathbf{g}_{z; \mathbf{x}}$ gets identified with γ .*

Remark 1.11. An alternative description of these **g**-vectors was given in [14, Theorem 10.2]. It is stated in different terms, and proved by a quite different method, relying on [10, Conjecture 7.12]. It is not difficult to check the equivalence of the two descriptions.

To give a formula for the F -polynomials, recall that, for each $i \in [1, n]$, there are one-parameter root subgroups in G given by

$$(1.14) \quad x_i(t) = \varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad x_{\bar{i}}(t) = \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

where $\varphi_i : SL_2 \rightarrow G$ denotes the canonical embedding corresponding to the simple root α_i .

Theorem 1.12. *Under the assumptions of Theorem 1.10, the F -polynomial $F_{z;\mathbf{x}}$ is given by*

$$(1.15) \quad F_{z;\mathbf{x}}(t_1, \dots, t_n) = \Delta_{\gamma, \gamma}(x_{\bar{1}}(1) \cdots x_{\bar{n}}(1) x_n(t_n) \cdots x_1(t_1)).$$

The following corollary agrees with [10, Conjecture 5.4].

Corollary 1.13. *Every F -polynomial in Theorem 1.12 has constant term 1.*

Example 1.14. Let c be as in Theorem 1.1 (for G of type A_n). Then we have $h(k; c) = n + 1 - k$ for $k = 1, \dots, n$. By Theorem 1.10, the cluster variable $x_{c^m \omega_k; c} = x_{[m+1, m+k]}$ has the \mathbf{g} -vector $c^m \omega_k = \omega_{m+k} - \omega_m$ (with the convention $\omega_0 = \omega_{n+1} = 0$). By Theorem 1.8, for $m \geq 1$, the denominator vector of $x_{c^m \omega_k; c}$ is equal to

$$c^{m-1} \omega_k - c^m \omega_k = \alpha_m + \alpha_{m+1} + \cdots + \alpha_{m+k-1}.$$

By computing the determinant in Theorem 1.12, we conclude that $x_{c^m \omega_k; c}$ has the F -polynomial

$$F(t_1, \dots, t_n) = 1 + t_m + t_m t_{m+1} + \cdots + t_m t_{m+1} \cdots t_{m+k-1}$$

(with the convention $t_0 = 0$).

The rest of the paper is devoted to the proofs of the above results. Most of our proofs rely on the following essentially combinatorial fact: every Coxeter element can be obtained from any other by a sequence of operations sending $c = s_{i_1} \cdots s_{i_n}$ to $\tilde{c} = s_{i_1} c s_{i_1}$. Thus to prove some statement for an arbitrary Coxeter element c , it is enough to check that it holds for some special choice of c , and that it is preserved by the above operations. As this special choice, we take a *bipartite* Coxeter element t , for which most of the above results were essentially established in earlier papers (see Section 2 for more details).

In Section 2, after a short reminder on root systems, we develop the combinatorics of the action of a Coxeter element on roots and fundamental weights, in particular proving Propositions 1.3, 1.6, and 1.7. In Section 3, we recall the necessary background on (reduced) double Bruhat cells and generalized minors, and prove the relations (1.10) and (1.11) from Theorem 1.5. Both sections 2 and 3 are totally independent of the theory of cluster algebras, which are not even mentioned there.

In Section 4 we recall basic definitions and facts about cluster algebras, following mainly [10], then prove Theorem 1.2. Note that some of the preliminary results given there (Propositions 4.5, 4.9 and 4.10) have not appeared in this form before, although they are easy consequences of the known results.

In Section 5 we prove Theorems 1.4 and 1.5. In the course of the proof we obtain some results of independent interest. Proposition 5.1 transfers the compatibility

degree function introduced in [8, Section 3] to each set $\Pi(c)$. This leads to a combinatorial description of all the clusters in $\mathcal{A}(c)$ given in Corollary 5.9. Another important result is Proposition 5.6, which provides, for every choice of a Coxeter element c , a realization of the cluster algebra with *universal coefficients* introduced in [10, Section 12], with respect to the initial seed with the exchange matrix $B(c)$. As a corollary of the technique developed in this section, we describe explicit isomorphisms between various algebras $\mathcal{A}(c)$ (Corollary 5.10) as well as corresponding geometric isomorphisms between various reduced double cells $L^{c,c^{-1}}$ (Remark 5.11).

In Section 6, we prove Theorems 1.8, 1.10 and 1.12, as well as Corollaries 1.9 and 1.13. As explained in Remark 6.1, these results allow us to give an alternative combinatorial description of the c -Cambrian fans studied in [14].

In the final Section 7, we illustrate the results of the paper by an example dealing with type A_n and the Coxeter element $c = s_1 \cdots s_n$ in the standard numbering of simple roots. In particular, we prove Theorem 1.1 and present an example making more explicit the construction given in Remark 5.11.

2. PROOFS OF PROPOSITIONS 1.3, 1.6, AND 1.7

We start by laying out the basic terminology and notation related to root systems, to be used throughout the paper; some of it has already appeared in the introduction. In what follows, $A = (a_{i,j})_{i,j \in I}$ is an indecomposable $n \times n$ Cartan matrix, i.e., one of the matrices A_n, B_n, \dots, G_2 in the Cartan-Killing classification (the general case in Propositions 1.3, 1.6, and 1.7 easily reduces to the case of A indecomposable).

Let Φ be the corresponding rank n root system with the set of simple roots $\{\alpha_i : i \in I\}$. Let W be the Weyl group of Φ , i.e., the group of linear transformations of the root space, generated by the *simple reflections* s_i ($i \in I$) whose action on simple roots is given by

$$(2.1) \quad s_i(\alpha_j) = \alpha_j - a_{i,j}\alpha_i.$$

Let $\{\omega_i : i \in I\}$ be the set of fundamental weights related to simple roots via

$$(2.2) \quad \alpha_j = \sum_{i \in I} a_{i,j}\omega_i.$$

The action of simple reflections on the fundamental weights is given by

$$(2.3) \quad s_i\omega_j = \begin{cases} \omega_i - \alpha_i & \text{if } j = i; \\ \omega_j & \text{if } j \neq i. \end{cases}$$

For future use, here are a couple of useful lemmas.

Lemma 2.1. *Suppose $I = \{1, \dots, n\}$ and $c = s_1 \cdots s_n$. For $i = 1, \dots, n$, let*

$$(2.4) \quad \beta_i = s_1 \cdots s_{i-1}\alpha_i,$$

so that $\{\beta_1, \dots, \beta_n\}$ is the set of positive roots β such that $c^{-1}\beta$ is negative. Then we have

$$(2.5) \quad \omega_i - c\omega_i = \beta_i,$$

and

$$(2.6) \quad \beta_j + \sum_{i=1}^{j-1} a_{i,j} \beta_i = \alpha_j.$$

Proof. The identity (2.5) is an immediate consequence of (2.3):

$$\beta_i = s_1 \cdots s_{i-1}(\omega_i - s_i \omega_i) = \omega_i - s_1 \cdots s_i \omega_i = \omega_i - c \omega_i.$$

As for (2.6), we have

$$\beta_j + \sum_{i=1}^{j-1} a_{i,j} \beta_i = s_1 \cdots s_{j-1} \alpha_j + \sum_{i=1}^{j-1} s_1 \cdots s_{i-1}(\alpha_j - s_i \alpha_j) = \alpha_j,$$

by telescoping. \square

Lemma 2.2. [4, Proposition VI.1.33] *In the situation of Lemma 2.1, let $\langle c \rangle$ denote the cyclic subgroup of W generated by c . Then every $\langle c \rangle$ -orbit in Φ contains exactly one positive root β such that $c^{-1}\beta$ is negative (i.e., one of the roots β_1, \dots, β_n) and exactly one positive root α such that $c\alpha$ is negative (i.e., one of the roots $-c^{-1}\beta_1, \dots, -c^{-1}\beta_n$).*

A *reduced word* for $w \in W$ is a sequence of indices $\mathbf{i} = (i_1, \dots, i_\ell)$ of shortest possible length $\ell = \ell(w)$ such that $w = s_{i_1} \cdots s_{i_\ell}$. The group W possesses a unique longest element denoted by w_\circ .

The *Coxeter graph* associated to Φ has the index set I as the set of vertices, with i and j joined by an edge whenever $a_{i,j}a_{j,i} > 0$. Since we assume that A is indecomposable, the root system Φ is irreducible, and the Coxeter graph I is a tree (one of the familiar Coxeter-Dynkin diagrams A_n, D_n, E_6, E_7, E_8).

Recall that a *Coxeter element* is the product of all simple reflections s_i (for $i \in I$) taken in an arbitrary order. Thus, we have $\ell(c) = n$. It is well-known that all Coxeter elements are conjugate to each other in W ; in particular, they have the same order h called the *Coxeter number* of W .

Note that Coxeter elements are in a natural bijection with orientations of the Coxeter graph: we orient an edge $j \rightarrow i$ if $i \prec_c j$, i.e., if i precedes j in some (equivalently, any) reduced word for c . We introduce the following elementary operation on Coxeter elements:

$$(2.7) \quad \begin{aligned} &\text{replace } c \text{ with } \tilde{c} \text{ if } c = s_i c' \text{ and } \tilde{c} = c' s_i \text{ for} \\ &\text{some } i \in I \text{ and } c' \in W \text{ with } \ell(c') = n - 1. \end{aligned}$$

We will use the following well-known fact:

$$(2.8) \quad \begin{aligned} &\text{any Coxeter element can be reached from} \\ &\text{any other one by a sequence of moves (2.7).} \end{aligned}$$

(Passing from Coxeter elements to corresponding orientations of the Coxeter graph, (2.8) translates into the statement that any orientation of a tree can be obtained from any other orientation by a repeated application of the following operation: reversing all arrows at some sink; this is easily proved by induction on the size of a tree.)

Our proofs of the needed properties of Coxeter elements will use the same strategy: show that the property in question is preserved under any operation (2.7), and prove that it holds for some particular choice of a Coxeter element. As this particular choice, we use the *bipartite* Coxeter element defined as follows.

Since the Coxeter graph is a tree, it is bipartite, so the set of vertices I is a disjoint union of two parts I_+ and I_- such that every edge joins two vertices from different parts. Note that I_+ and I_- are determined uniquely up to renaming. We write $\varepsilon(i) = \varepsilon$ for $i \in I_\varepsilon$.

Now we define the bipartite Coxeter element t by setting

$$(2.9) \quad t = t_+ t_- ,$$

where

$$(2.10) \quad t_\pm = \prod_{\varepsilon(i)=\pm 1} s_i .$$

Note that the order of factors in (2.10) does not matter because s_i and s_j commute whenever $\varepsilon(i) = \varepsilon(j)$. Let \mathbf{i}_- and \mathbf{i}_+ be some reduced words for the elements t_- and t_+ , respectively.

Lemma 2.3. [4, Exercise V.6.2] *We have $\underbrace{t_+ t_- t_+ \cdots t_\pm t_\mp}_{h \text{ factors}} = w_\circ$. Moreover, the word*

$$(2.11) \quad \mathbf{i}_\circ \stackrel{\text{def}}{=} \underbrace{\mathbf{i}_+ \mathbf{i}_- \mathbf{i}_+ \cdots \mathbf{i}_\pm \mathbf{i}_\mp}_h$$

(concatenation of h segments) is a reduced word for w_\circ .

Regarding Lemma 2.3, recall from the tables in [4] that h is even for all types except A_n with n even; in the exceptional case of type A_{2e} , we have $h = 2e + 1$. If h is even, then Lemma 2.3 says that $t^{h/2} = w_\circ$.

Now everything is in place for the proofs of Propositions 1.3, 1.6, and 1.7. We start by observing that Proposition 1.3 is a consequence of the following seemingly weaker statement:

$$(2.12) \quad \begin{array}{l} \text{for every Coxeter element } c \text{ and } i \in I, \text{ the} \\ \text{weight } w_\circ \omega_i \text{ belongs to the } \langle c \rangle\text{-orbit of } \omega_i. \end{array}$$

Indeed, assuming (2.12), we can define $h(i; c)$ as the smallest positive integer n such that $c^m \omega_i = w_\circ \omega_i$. To prove the inequalities in (1.8), note that, in view of (2.5), we have $c^m \omega_i - c^{m+1} \omega_i = c^m \beta_i$ for $m \in \mathbb{Z}$, and so the differences $c^m \omega_i - c^{m+1} \omega_i$ form the $\langle c \rangle$ -orbit of β_i in Φ . Now recall Lemma 2.2. Since ω_i is the maximal element of its W -orbit, while $w_\circ \omega_i$ is the minimal element of the same orbit, we see that $\beta = \omega_i - c \omega_i$ (resp. $\alpha = c^{-1} w_\circ \omega_i - w_\circ \omega_i$) is the unique positive root in the $\langle c \rangle$ -orbit of β_i such that $c^{-1} \beta$ (resp. $c \alpha$) is negative. It follows that all the roots $\gamma = c^m \omega_i - c^{m+1} \omega_i$ with $0 \leq m < h(i; c)$ are positive, as required.

As an easy consequence of Lemma 2.3, the statement (2.12) (hence Proposition 1.3) holds for the bipartite Coxeter element t ; furthermore, we have

$$(2.13) \quad h(i; t) = \begin{cases} \lfloor \frac{h}{2} \rfloor & \text{for } \varepsilon(i) = -1 ; \\ \lceil \frac{h}{2} \rceil & \text{for } \varepsilon(i) = +1 . \end{cases}$$

The fact that t satisfies Propositions 1.6 and 1.7, follows at once from (2.13) together with an observation that, for every $i \in I$, we have

$$(2.14) \quad \varepsilon(i^*) = \begin{cases} \varepsilon(i) & \text{if } h \text{ is even;} \\ -\varepsilon(i) & \text{if } h \text{ is odd.} \end{cases}$$

In view of (2.8), to finish the proofs of Propositions 1.3, 1.6, and 1.7, it suffices to assume that they hold for some Coxeter element c and show that the same is true for the element \tilde{c} obtained from c via (2.7). Without loss of generality, we assume that $I = \{1, \dots, n\}$, $c = s_1 s_2 \cdots s_n$, and $\tilde{c} = s_2 \cdots s_n s_1$.

The key statement to prove is the following: \tilde{c} satisfies (2.12) (hence Proposition 1.3), and we have

$$(2.15) \quad h(i; \tilde{c}) = \begin{cases} h(i; c) - 1 & \text{if } i = 1 \text{ and } i^* \neq 1; \\ h(i; c) + 1 & \text{if } i \neq 1 \text{ and } i^* = 1; \\ h(i; c) & \text{otherwise.} \end{cases}$$

We split the proof of (2.15) into four cases:

Case 1. Let $i \neq 1$ and $i^* \neq 1$. Since $\tilde{c}^m = s_1 c^m s_1$ for all $m > 0$, using (2.3), we obtain

$$\tilde{c}^m \omega_i = -\omega_{i^*} \iff s_1 c^m s_1 \omega_i = -\omega_{i^*} \iff c^m \omega_i = -\omega_{i^*},$$

implying that $h(i; \tilde{c}) = h(i; c)$.

Case 2. Let $i = i^* = 1$. Writing $\tilde{c}^m = s_2 \cdots s_n c^m s_n \cdots s_2$ and using (2.3), we obtain

$$\tilde{c}^m \omega_1 = -\omega_1 \iff c^m \omega_1 = -\omega_1,$$

again implying that $h(1; \tilde{c}) = h(1; c)$.

Case 3. Let $i = 1$ and $i^* \neq 1$. Writing $\tilde{c}^m = s_1 c^{m+1} s_n \cdots s_2$ and using (2.3), we obtain

$$\tilde{c}^m \omega_1 = -\omega_{i^*} \iff c^{m+1} \omega_1 = -\omega_{i^*},$$

implying that $h(i; \tilde{c}) = h(i; c) - 1$.

Case 4. Finally, let $i \neq 1$ and $i^* = 1$. Writing $\tilde{c}^m = s_2 \cdots s_n c^{m-1} s_1$ and using (2.3), we obtain

$$\tilde{c}^m \omega_i = -\omega_1 \iff c^{m-1} \omega_i = -\omega_1,$$

implying that $h(i; \tilde{c}) = h(i; c) + 1$.

This concludes the proof of Proposition 1.3 and the relation (2.15).

The proofs of Propositions 1.6 and 1.7 now become purely combinatorial exercises. Namely, let again $c = s_1 s_2 \cdots s_n$ and $\tilde{c} = s_2 \cdots s_n s_1$. To prove Proposition 1.7, it suffices to show that $h(i; \tilde{c}) + h(i^*; \tilde{c}) = h(i; c) + h(i^*; c)$ for all i , which is immediate from (2.15).

To prove Proposition 1.6, it suffices to show the following: if the numbers $h(i; c)$ satisfy (1.12), and the $h(i; \tilde{c})$ are given by (2.15), then the $h(i; \tilde{c})$ also satisfy (1.12) with c replaced by \tilde{c} . There are several cases to consider.

Case 1. Suppose that $i \prec_{\tilde{c}} j$, and $i^* \prec_{\tilde{c}} j^*$. Then we have $i \neq 1$ and $i^* \neq 1$, hence $h(i; \tilde{c}) = h(i; c)$. This case breaks into four subcases according to which of the two indices j and j^* are equal to 1. In each of these subcases, the desired equality

$h(j; \tilde{c}) = h(i; \tilde{c})$ is seen by a direct inspection. For instance, if $j = 1 \neq j^*$, we have $h(j; \tilde{c}) = h(j; c) - 1$ by (2.15); on the other hand, in this case we have $j \prec_c i$, and $i^* \prec_c j^*$, hence $h(j; c) - 1 = h(i; c)$ by (1.12).

Case 2. Now suppose that $i \prec_{\tilde{c}} j$, and $j^* \prec_{\tilde{c}} i^*$. Then we have $i \neq 1$ and $j^* \neq 1$. This case also breaks into four subcases according to which of the two indices j and i^* are equal to 1. Again, in each of these subcases, the desired equality $h(j; \tilde{c}) = h(i; \tilde{c}) - 1$ is seen by a direct inspection. For instance, if $j = 1 \neq i^*$, we have $h(i; \tilde{c}) = h(i; c)$, and $h(j; \tilde{c}) = h(j; c) - 1$ by (2.15); on the other hand, in this case we have $j \prec_c i$, and $j^* \prec_c i^*$, hence $h(j; c) = h(i; c)$ by (1.12).

This concludes the proofs of Propositions 1.3, 1.6, and 1.7.

3. PROOFS OF RELATIONS (1.10) AND (1.11)

For the convenience of the reader, we start by briefly recalling necessary facts about generalized minors and reduced double cells; more details can be found in [5, 3].

Let \mathfrak{g} be a complex semisimple Lie algebra of rank n with Chevalley generators f_i, α_i^\vee , and e_i for $i \in I$, where I is an n -element index set, which will be often identified with $\{1, \dots, n\}$. The elements α_i^\vee are *simple coroots* of \mathfrak{g} ; they form a basis of a Cartan subalgebra \mathfrak{h} of \mathfrak{g} . The *simple roots* α_i ($i \in I$) form a basis in the dual space \mathfrak{h}^* such that $[h, e_i] = \alpha_i(h)e_i$, and $[h, f_i] = -\alpha_i(h)f_i$ for any $h \in \mathfrak{h}$ and $i \in I$. The structure of \mathfrak{g} is uniquely determined by the *Cartan matrix* $A = (a_{i,j})$ given by $a_{ij} = \alpha_j(\alpha_i^\vee)$.

Let G be the simply connected complex semisimple Lie group with the Lie algebra \mathfrak{g} . For every $i \in I$, let $\varphi_i : SL_2 \rightarrow G$ denote the canonical embedding corresponding to the simple root α_i . We use the notation

$$(3.1) \quad x_i(t) = \varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \exp(te_i), \quad x_{\bar{i}}(t) = \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \exp(tf_i).$$

We also set

$$t^{\alpha_i^\vee} = \varphi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in H$$

for any $i \in I$ and any $t \neq 0$. Let N (resp. N_-) be the maximal unipotent subgroup of G generated by all $x_i(t)$ (resp. $x_{\bar{i}}(t)$). Let H be the maximal torus in G with the Lie algebra \mathfrak{h} . Let $B = HN$ and $B_- = HN_-$ be the corresponding two opposite Borel subgroups.

The Weyl group W associated to A is naturally identified with $\text{Norm}_G(H)/H$; this identification sends each simple reflection s_i to the coset $\overline{s_i}H$, where the representative $\overline{s_i} \in \text{Norm}_G(H)$ is defined by

$$\overline{s_i} = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The elements $\overline{s_i}$ satisfy the braid relations in W ; thus the representative \overline{w} can be unambiguously defined for any $w \in W$ by requiring that $\overline{uv} = \overline{u} \cdot \overline{v}$ whenever $\ell(uv) = \ell(u) + \ell(v)$.

With some abuse of notation, we identify the weight lattice P in \mathfrak{h}^* with the group of rational multiplicative characters of H , here written in the exponential notation:

a weight $\gamma \in P$ acts by $a \mapsto a^\gamma$. Under this identification, the fundamental weights $\omega_1, \dots, \omega_n$ act in H by $(t^{\alpha_j^\vee})^{\omega_i} = t^{\delta_{ij}}$.

Recall that the set $G_0 = N_-HN$ of elements $x \in G$ that have Gaussian decomposition is open and dense in G ; this (unique) decomposition of $x \in N_-HN$ is written as $x = [x]_-[x]_0[x]_+$.

We now recall some basic properties of *generalized minors* introduced in [5]. For $u, v \in W$ and $i \in I$, the *generalized minor* $\Delta_{u\omega_i, v\omega_i}$ is the regular function on G whose restriction to the open set $\bar{u}G_0\bar{v}^{-1}$ is given by

$$(3.2) \quad \Delta_{u\omega_i, v\omega_i}(x) = ([\bar{u}^{-1}x\bar{v}]_0)^{\omega_i}.$$

As shown in [5], $\Delta_{u\omega_i, v\omega_i}$ depends on the weights $u\omega_i$ and $v\omega_i$ alone, not on the particular choice of u and v . In the special case $G = SL_{n+1}$, the generalized minors are nothing but the ordinary minors of a matrix.

Generalized minors have the following properties (see [5, (2.14), (2.25)]):

$$(3.3) \quad \Delta_{\gamma, \delta}(a_1 x a_2) = a_1^\gamma a_2^\delta \Delta_{\gamma, \delta}(x) \quad (a_1, a_2 \in H; x \in G);$$

$$(3.4) \quad \Delta_{\gamma, \delta}(x) = \Delta_{\delta, \gamma}(x^T),$$

where $x \mapsto x^T$ is the “transpose” involutive antiautomorphism of G acting on generators by

$$(3.5) \quad a^T = a \quad (a \in H), \quad x_i(t)^T = x_{\bar{i}}(t), \quad x_{\bar{i}}(t)^T = x_i(t).$$

We will also use the involutive antiautomorphism ι of G introduced in [5, (2.2)]; it is defined by

$$(3.6) \quad a^\iota = a^{-1} \quad (a \in H), \quad x_i(t)^\iota = x_i(t), \quad x_{\bar{i}}(t)^\iota = x_{\bar{i}}(t).$$

By [5, (2.25)], we have

$$(3.7) \quad \Delta_{\gamma, \delta}(x) = \Delta_{-\delta, -\gamma}(x^\iota)$$

for any generalized minor $\Delta_{\gamma, \delta}$, and any $x \in G$.

The following important identity was obtained in [5, Theorem 1.17].

Proposition 3.1. *Suppose $u, v \in W$ and $k \in I$ are such that $us_k\omega_k < u\omega_k$ and $vs_k\omega_k < v\omega_k$. Then*

$$(3.8) \quad \Delta_{u\omega_k, v\omega_k} \Delta_{us_k\omega_k, vs_k\omega_k} = \Delta_{us_k\omega_k, v\omega_k} \Delta_{u\omega_k, vs_k\omega_k} + \prod_{i \neq k} \Delta_{u\omega_i, v\omega_i}^{-a_{i,k}}.$$

The group G has two *Bruhat decompositions*, with respect to opposite Borel subgroups B and B_- :

$$G = \bigcup_{u \in W} BuB = \bigcup_{v \in W} B_-vB_-.$$

The *double Bruhat cells* $G^{u,v}$ are defined by $G^{u,v} = BuB \cap B_-vB_-$. These varieties were introduced and studied in [5]. Each double Bruhat cell can be defined inside G by a collection of vanishing/non-vanishing conditions of the form $\Delta(x) = 0$ and $\Delta(x) \neq 0$, where Δ is a generalized minor. The following description can be found in [2, Proposition 2.8].

Proposition 3.2. *A double Bruhat cell $G^{u,v}$ is given inside G by the following conditions, for all $i \in I$:*

$$(3.9) \quad \Delta_{u'\omega_i, \omega_i} = 0 \quad \text{whenever } u'\omega_i \not\leq u\omega_i \text{ in the Bruhat order,}$$

$$(3.10) \quad \Delta_{\omega_i, v'\omega_i} = 0 \quad \text{whenever } v'\omega_i \not\leq v^{-1}\omega_i \text{ in the Bruhat order,}$$

$$(3.11) \quad \Delta_{u\omega_i, \omega_i} \neq 0, \quad \Delta_{\omega_i, v^{-1}\omega_i} \neq 0.$$

In this paper we concentrate on the following subset $L^{u,v} \subset G^{u,v}$ introduced in [3] and called a *reduced double Bruhat cell*:

$$(3.12) \quad L^{u,v} = N\bar{u}N \cap B_-vB_-.$$

The equations defining $L^{u,v}$ inside $G^{u,v}$ look as follows.

Proposition 3.3. [3, Proposition 4.3] *An element $x \in G^{u,v}$ belongs to $L^{u,v}$ if and only if $\Delta_{u\omega_i, \omega_i}(x) = 1$ for $i \in [1, n]$.*

The variety $L^{u,v}$ (resp. $G^{u,v}$) is biregularly isomorphic to a Zariski open subset of an affine space of dimension $\ell(u) + \ell(v)$ (resp. $\ell(u) + \ell(v) + n$). Local coordinates in $L^{u,v}$ and $G^{u,v}$ were constructed in [5, 3]. To simplify the notation, we will describe them only in the following special case.

In the rest of the section we assume that the index set I is $\{1, \dots, n\}$, and work with the fixed Coxeter element $c = s_1 \cdots s_n$, and with the varieties $L^{c, c^{-1}}$ and $G^{c, c^{-1}}$.

In accordance with [5, Theorem 1.2], a generic element $x \in G^{c, c^{-1}}$ can be uniquely factored as

$$(3.13) \quad x = x_{\bar{1}}(u_1) \cdots x_{\bar{n}}(u_n) a x_n(t_n) \cdots x_1(t_1)$$

for some $a \in H$ and $u_i, t_i \in \mathbb{C}^*$ (see (3.1)).

In the case of the reduced double cell $L^{c, c^{-1}}$, the factorization (3.13) can be modified as follows. For any nonzero $u \in \mathbb{C}$ and any $i = 1, \dots, n$, denote

$$(3.14) \quad x_{-i}(u) = x_{\bar{i}}(u)u^{-\alpha_i^\vee} = \varphi_i \begin{pmatrix} u^{-1} & 0 \\ 1 & u \end{pmatrix}.$$

In accordance with [3, Proposition 4.5], a generic element $x \in L^{c, c^{-1}}$ can be uniquely factored as

$$(3.15) \quad x = x_{-1}(u_1) \cdots x_{-n}(u_n) x_n(t_n) \cdots x_1(t_1),$$

where $u_i, t_i \in \mathbb{C}^*$.

After this preparation, let us turn to the proof of (1.11). Let $k = 1, \dots, n$ and $1 \leq m \leq h(k; c)$. Specializing (3.8) for $u = v = c^{m-1}s_1 \cdots s_{k-1}$ (and taking into account (2.3)), we get

$$(3.16) \quad \begin{aligned} \Delta_{c^{m-1}\omega_k, c^{m-1}\omega_k} \Delta_{c^m\omega_k, c^m\omega_k} &= \Delta_{c^m\omega_k, c^{m-1}\omega_k} \Delta_{c^{m-1}\omega_k, c^m\omega_k} \\ &+ \prod_{i < k} \Delta_{c^m\omega_i, c^m\omega_i}^{-a_{i,k}} \prod_{i > k} \Delta_{c^{m-1}\omega_i, c^{m-1}\omega_i}^{-a_{i,k}}. \end{aligned}$$

Remembering (1.5) and (1.6), and comparing (3.16) with the desired equality (1.11), we see that it remains to prove the following lemma.

Lemma 3.4. *For every $k = 1, \dots, n$ and $1 \leq m \leq h(k; c)$, the restriction to $L^{c, c^{-1}}$ of the function $\Delta_{c^m \omega_k, c^{m-1} \omega_k} \Delta_{c^{m-1} \omega_k, c^m \omega_k}$ is equal to that of the product*

$$(3.17) \quad \prod_{j=1}^n (\Delta_{\omega_j, c \omega_j} \prod_{i < j} \Delta_{\omega_i, c \omega_i}^{a_{i,j}})^{[c^{m-1} \omega_k - c^m \omega_k : \alpha_j]}.$$

Proof. First of all, in view of (2.5), we have

$$(3.18) \quad c^{m-1} \omega_k - c^m \omega_k = c^{m-1} \beta_k.$$

Using (2.6), we can rewrite the product in (3.17) as

$$(3.19) \quad \prod_{j=1}^n (\Delta_{\omega_j, c \omega_j} \prod_{i < j} \Delta_{\omega_i, c \omega_i}^{a_{i,j}})^{[c^{m-1} \omega_k - c^m \omega_k : \alpha_j]} = \prod_{i=1}^n \Delta_{\omega_i, c \omega_i}^{[c^{m-1} \beta_k : \beta_i]},$$

where the exponent $[c^{m-1} \beta_k : \beta_i]$ stands for the coefficient of β_i in the expansion of $c^{m-1} \beta_k$ in the basis $\{\beta_1, \dots, \beta_n\}$.

Now let us deal with the functions $\Delta_{c^m \omega_k, c^{m-1} \omega_k}$ and $\Delta_{c^{m-1} \omega_k, c^m \omega_k}$. Let x be a generic element of $G^{c, c^{-1}}$ expressed as in (3.13). We claim that

$$(3.20) \quad \Delta_{c^{m-1} \omega_k, c^m \omega_k}(x) = a^{c^{m-1} \omega_k} \prod_{j=1}^n t_j^{[c^{m-1} \beta_k : \alpha_j]}.$$

Our proof of (3.20) uses a little representation theory. Consider the ring of regular functions $\mathbb{C}[G]$ as a $G \times G$ -representation under the action $(g_1, g_2)f(x) = f(g_1^T x g_2)$ (see (3.5) for the definition of g_1^T). We denote by $f \mapsto (v_1, v_2)f$ the corresponding action of $U(\mathfrak{g}) \times U(\mathfrak{g})$, where $U(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g} . For every $f \in \mathbb{C}[G]$ and $x \in G^{c, c^{-1}}$ as above, $f(x)$ is a polynomial in u_i and t_i , and the coefficient of each monomial $u_1^{h_1} \dots u_n^{h_n} t_1^{d_1} \dots t_n^{d_n}$ is equal to $((e_1^{(h_1)} \dots e_n^{(h_n)}, e_n^{(d_n)} \dots e_1^{(d_1)})f)(a)$, where $e_i^{(d)}$ stands for the divided power $e_i^d/d!$ (cf. [1, Lemma 3.7.5]). Furthermore, if f is a weight vector of weight (γ, γ') then $f(a)$ can be nonzero only if $\gamma = \gamma'$.

Now recall that, in view of (3.3), each generalized minor $\Delta_{\gamma, \delta}$ with $\gamma, \delta \in W\omega_k$ is a weight vector of weight (γ, δ) in the $G \times G$ -irreducible representation with highest weight (ω_k, ω_k) . Recall also the following standard facts from the representation theory of G : the weight polytope of the fundamental G -representation V_{ω_k} with highest weight ω_k has $W\omega_k$ as the set of vertices, and for each weight $\delta \in W\omega_k$, the corresponding weight subspace $V_{\omega_k}(\delta)$ is one-dimensional. Together with basic facts about the representations of SL_2 , this implies the following: if $\delta \in W\omega_k$, and $s_i \delta \geq \delta$ for some i then $s_i \delta - \delta = d_i \alpha_i$, where $e_i^{(d)}(V_{\omega_k}(\delta)) = \{0\}$ for $d > d_i$; furthermore, $e_i^{(d_i)}$ establishes the same isomorphism $V_{\omega_k}(\delta) \rightarrow V_{\omega_k}(s_i \delta)$ as the group element $\bar{s}_i^{-1} \in G$.

Applying all this to $\gamma = c^{m-1} \omega_k$ and $\delta = c^m \omega_k$ with $1 \leq m \leq h(k; c)$, we conclude that the only tuple of nonnegative integers $(h_1, \dots, h_n; d_1, \dots, d_n)$ such that $(e_1^{(h_1)} \dots e_n^{(h_n)}, e_n^{(d_n)} \dots e_1^{(d_1)})(\Delta_{\gamma, \delta}) \neq 0$ is the one with all h_i equal to 0, and $\sum_i d_i \alpha_i = \gamma - \delta$. Furthermore, for this tuple we have

$$((e_1^{(h_1)} \dots e_n^{(h_n)}, e_n^{(d_n)} \dots e_1^{(d_1)})\Delta_{\gamma, \delta})(a) = \Delta_{\gamma, \gamma}(a) = a^\gamma,$$

implying (3.20).

We now show that

$$(3.21) \quad \Delta_{c^{m-1}\omega_k, c^m\omega_k}(x) = \prod_{i=1}^n \Delta_{\omega_i, c\omega_i}(x)^{[c^{m-1}\beta_k : \beta_i]}$$

for any $x \in G^{c, c^{-1}}$. It is enough to show that (3.21) holds for x of the form (3.13). Then both sides of (3.21) are given by (3.20). Comparing the contributions of $a \in H$ and of each t_j on both sides, we only have to prove the following two identities:

$$(3.22) \quad c^{m-1}\omega_k = \sum_{i=1}^n [c^{m-1}\beta_k : \beta_i]\omega_i;$$

$$(3.23) \quad [c^{m-1}\beta_k : \alpha_j] = \sum_{i=1}^n [c^{m-1}\beta_k : \beta_i][\beta_i : \alpha_j].$$

The identity (3.23) is immediate, so let us prove (3.22). Abbreviating $[c^{m-1}\beta_k : \beta_i] = b_i$ and recalling (3.18), we can rewrite the equality

$$c^{m-1}\beta_k = \sum_{i=1}^n b_i\beta_i$$

in the form

$$c^{m-1}\omega_k - \sum_{i=1}^n b_i\omega_i = c(c^{m-1}\omega_k - \sum_{i=1}^n b_i\omega_i).$$

To finish the proof of (3.22), it remains to use the well-known property that no Coxeter element has 1 as an eigenvalue (see, e.g., [12, Lemma 8.1]).

In view of (3.21) and (3.19), to finish the proof of Lemma 3.4 (hence that of the relation (1.11)), it remains to show that the restriction to $L^{c, c^{-1}}$ of each $\Delta_{c^m\omega_k, c^{m-1}\omega_k}$ is equal to 1. In view of (3.4), the identity (3.21) implies that

$$\Delta_{c^m\omega_k, c^{m-1}\omega_k}(x) = \prod_{i=1}^n \Delta_{c\omega_i, \omega_i}(x)^{[c^{m-1}\beta_k : \beta_i]}$$

for any $x \in G^{c, c^{-1}}$. Now the fact that the right-hand side of this equality is equal to 1 for $x \in L^{c, c^{-1}}$, follows from Proposition 3.3, and we are done. \square

Let us now turn to the proof of (1.10). It is enough to show that both sides of this identity take the same values at a general element $x \in L^{c, c^{-1}}$ expressed in the form (3.15).

To compute $\Delta_{\omega_i, \omega_i}(x) = ([x]_0)^{\omega_i}$ (see (3.2)), express each factor $x_{-j}(u_j)$ in x as $x_{\bar{j}}(u_j)u_j^{-\alpha_j^\vee}$, and move the factors $u_j^{-\alpha_j^\vee}$ all the way to the “center”, using the commutation relations in [5, (2.5)]:

$$(3.24) \quad ax_j(t) = x_j(a^{\alpha_j}t)a, \quad ax_{\bar{j}}(u) = x_{\bar{j}}(a^{-\alpha_j}u)a \quad (a \in H).$$

We obtain

$$x = x_{\bar{1}}(r_1) \cdots x_{\bar{n}}(r_n) \left(\prod_{j=1}^n u_j^{-\alpha_j^\vee} \right) x_n(t_n) \cdots x_1(t_1),$$

where the precise expression for r_j is immaterial for our current purposes. It follows that $[x]_0 = \prod_{j=1}^n u_j^{-\alpha_j^\vee}$, hence

$$(3.25) \quad \Delta_{\omega_i, \omega_i}(x) = \prod_{j=1}^n u_j^{\omega_i(-\alpha_j^\vee)} = u_i^{-1}.$$

In view of (3.20), we also have

$$\Delta_{\omega_i, c\omega_i}(x) = u_i^{-1} \prod_{j=1}^n t_j^{[\beta_i; \alpha_j]}.$$

Remembering the definition (1.6) and using (2.6), we get

$$(3.26) \quad y_{k;c}(x) = \Delta_{\omega_k, c\omega_k}(x) \prod_{i < k} \Delta_{\omega_i, c\omega_i}(x)^{a_{i,k}} = t_k u_k^{-1} \prod_{j < k} u_j^{-a_{j,k}}.$$

Substituting the expressions in (3.25) and (3.26) into (1.10), we can simplify it as follows:

$$(3.27) \quad \Delta_{-\omega_k, -\omega_k}(x) = t_k \prod_{j > k} \Delta_{-\omega_j, -\omega_j}(x)^{-a_{j,k}} + u_k.$$

To prove (3.27), note that, according to (3.7), we have

$$\Delta_{-\omega_k, -\omega_k}(x) = \Delta_{\omega_k, \omega_k}(x^\iota).$$

Here $x \mapsto x^\iota$ is an involutive antiautomorphism of the group G defined in (3.6). As an easy consequence of this definition, we have

$$x_{-j}(u)^\iota = x_{-j}(u^{-1}),$$

hence

$$x^\iota = x_1(t_1) \cdots x_n(t_n) x_{-n}(u_n^{-1}) \cdots x_{-1}(u_1^{-1}).$$

To compute $\Delta_{\omega_k, \omega_k}(x^\iota)$ we use the commutation relations in [3, Proposition 7.2], which we rewrite as follows:

$$x_j(t) x_{-i}(u^{-1}) = x_{-i}(u^{-1}) x_j(t u^{-a_{i,j}}) \quad (i \neq j),$$

$$x_j(t) x_{-j}(u^{-1}) = x_{-j}((t + u)^{-1}) x_j(w),$$

where the precise expression for w is immaterial for our purposes. Iterating these relations, we can rewrite x^ι in the form

$$x^\iota = x_{-n}(v_n^{-1}) \cdots x_{-1}(v_1^{-1}) x_1(w_1) \cdots x_n(w_n),$$

where the parameters v_1, \dots, v_n satisfy the recurrence relations

$$v_k = t_k \prod_{j > k} v_j^{-a_{j,k}} + u_k.$$

It remains to observe that $\Delta_{\omega_k, \omega_k}(x^\iota) = v_k$ (cf. (3.25)), so the last formula implies (3.27). This completes the proof of (1.11).

4. PROOF OF THEOREM 1.2

We start by recalling basic definitions and facts on cluster algebras following mostly [10]. The definition of a cluster algebra \mathcal{A} starts with introducing its ground ring. Let \mathbb{P} be a *semifield*, i.e., an abelian multiplicative group endowed with a binary operation of (*auxiliary*) *addition* \oplus which is commutative, associative, and distributive with respect to the multiplication in \mathbb{P} . The multiplicative group of \mathbb{P} is torsion-free [7, Section 5], hence its group ring $\mathbb{Z}\mathbb{P}$ —which will be used as a *ground ring* for \mathcal{A} —is a domain. Let $\mathbb{Q}(\mathbb{P})$ denote the field of fractions of $\mathbb{Z}\mathbb{P}$.

Every finite family $\{u_j : j \in J\}$ gives rise to a *tropical semifield* $\text{Trop}(u_j : j \in J)$ defined as follows. As a multiplicative group, $\text{Trop}(u_j : j \in J)$ is an abelian group freely generated by the elements u_j ($j \in J$); and the addition \oplus in $\text{Trop}(u_j : j \in J)$ is given by

$$(4.1) \quad \prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)}.$$

Thus, the group ring of $\text{Trop}(u_j : j \in J)$ is the ring of Laurent polynomials in the variables u_j . A cluster algebra associated to a tropical semifield is said to be of *geometric type*.

As an *ambient field* for a cluster algebra \mathcal{A} of rank n , we take a field \mathcal{F} isomorphic to the field of rational functions in n independent variables, with coefficients in $\mathbb{Q}(\mathbb{P})$. Note that the definition of \mathcal{F} ignores the auxiliary addition in \mathbb{P} .

Definition 4.1 (*Seeds*). A (labeled) *seed* in \mathcal{F} is a triple $(\mathbf{x}, \mathbf{y}, B)$, where

- $\mathbf{x} = \{x_i : i \in I\}$ is an n -tuple of elements of \mathcal{F} forming a *free generating set* for \mathcal{F} , that is, the elements x_i for $i \in I$ are algebraically independent over $\mathbb{Q}(\mathbb{P})$, and $\mathcal{F} = \mathbb{Q}(\mathbb{P})(\mathbf{x})$.
- $\mathbf{y} = \{y_i : i \in I\}$ is an n -tuple of elements of \mathbb{P} .
- $B = (b_{i,j})_{i,j \in I}$ is an $n \times n$ integer matrix which is *skew-symmetrizable*, that is, $d_i b_{i,j} = -d_j b_{j,i}$ for some positive integers d_i ($i \in I$).

We refer to \mathbf{x} as the *cluster* of a seed $(\mathbf{x}, \mathbf{y}, B)$, to the tuple \mathbf{y} as the *coefficient tuple*, and to the matrix B as the *exchange matrix*.

We will use the notation $[b]_+ = \max(b, 0)$.

Definition 4.2 (*Seed mutations*). Let $(\mathbf{x}, \mathbf{y}, B)$ be a seed in \mathcal{F} , and $k \in I$. The *seed mutation* μ_k in direction k transforms $(\mathbf{x}, \mathbf{y}, B)$ into the seed $\mu_k(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}', \mathbf{y}', B')$ defined as follows:

- The entries of $B' = (b'_{i,j})$ are given by

$$(4.2) \quad b'_{i,j} = \begin{cases} -b_{i,j} & \text{if } i = k \text{ or } j = k; \\ b_{i,j} + [b_{ik}]_+ [b_{kj}]_+ - [-b_{i,k}]_+ [-b_{k,j}]_+ & \text{otherwise.} \end{cases}$$

- The coefficient tuple \mathbf{y}' is given by

$$(4.3) \quad y'_j = \begin{cases} y_k^{-1} & \text{if } j = k; \\ y_j y_k^{[b_{k,j}]_+} (y_k \oplus 1)^{-b_{k,j}} & \text{if } j \neq k. \end{cases}$$

- The cluster \mathbf{x}' is given by $x'_j = x_j$ for $j \neq k$, whereas $x'_k \in \mathcal{F}$ is determined by the *exchange relation*

$$(4.4) \quad x'_k = \frac{y_k \prod x_i^{[b_{i,k}]_+} + \prod x_i^{[-b_{i,k}]_+}}{(y_k \oplus 1)x_k}.$$

It is easy to see that B' is skew-symmetrizable (with the same choice of the factors d_i), implying that $(\mathbf{x}', \mathbf{y}', B')$ is indeed a seed. Furthermore, the seed mutation μ_k is involutive, that is, it transforms $(\mathbf{x}', \mathbf{y}', B')$ back into $(\mathbf{x}, \mathbf{y}, B)$. This makes the following equivalence relation on seeds well defined: we say that two seeds $(\mathbf{x}, \mathbf{y}, B)$ and $(\mathbf{x}', \mathbf{y}', B')$ are *mutation equivalent*, and write $(\mathbf{x}, \mathbf{y}, B) \sim (\mathbf{x}', \mathbf{y}', B')$, if $(\mathbf{x}', \mathbf{y}', B')$ can be obtained from $(\mathbf{x}, \mathbf{y}, B)$ by a sequence of seed mutations. For a mutation equivalence class of seeds \mathcal{S} , we denote by $\mathcal{X} = \mathcal{X}(\mathcal{S})$ the union of clusters of all seeds in \mathcal{S} . We refer to the elements in \mathcal{X} as *cluster variables*.

Definition 4.3 (*Cluster algebra*). The *cluster algebra* \mathcal{A} associated with a mutation equivalence class of seeds \mathcal{S} is the $\mathbb{Z}\mathbb{P}$ -subalgebra of the ambient field \mathcal{F} generated by all cluster variables: $\mathcal{A} = \mathbb{Z}\mathbb{P}[\mathcal{X}]$. We denote $\mathcal{A} = \mathcal{A}(\mathcal{S}) = \mathcal{A}(\mathbf{x}, \mathbf{y}, B) = \mathcal{A}(\mathbf{y}, B)$, where $(\mathbf{x}, \mathbf{y}, B)$ is any seed in \mathcal{S} .

Following [10], we now introduce an important special system of coefficients called *principal*.

Definition 4.4 (*Principal coefficients*). We say that a seed $(\mathbf{x}^\circ, \mathbf{y}^\circ, B)$ has *principal coefficients* if the coefficient semifield \mathbb{P}° is the tropical semifield $\mathbb{P}^\circ = \text{Trop}(y_i^\circ : i \in I)$ generated by the coefficient tuple \mathbf{y}° .

Let $\mathcal{A}^\circ = \mathcal{A}(\mathbf{x}^\circ, \mathbf{y}^\circ, B)$ be the cluster algebra with principal coefficients at an initial seed $(\mathbf{x}^\circ, \mathbf{y}^\circ, B)$. For convenience, let us use $I = \{1, \dots, n\}$ as the set of indices for the initial cluster variables $x_1^\circ, \dots, x_n^\circ$ and the coefficient tuple $y_1^\circ, \dots, y_n^\circ$. For $j = 1, \dots, n$, we set

$$(4.5) \quad \widehat{y}_j^\circ = y_j^\circ \prod_{i=1}^n (x_i^\circ)^{b_{i,j}} \in \mathcal{F}.$$

As a special case of [10, Corollary 6.3], every cluster variable $z \in \mathcal{A}^\circ$ can be (uniquely) written in the form

$$(4.6) \quad z = (x_1^\circ)^{g_1} \cdots (x_n^\circ)^{g_n} F_{z; \mathbf{x}^\circ}(\widehat{y}_1^\circ, \dots, \widehat{y}_n^\circ)$$

for some integer vector $\mathbf{g}_{z; \mathbf{x}^\circ} = (g_1, \dots, g_n) \in \mathbb{Z}^n$, and some integer polynomial $F_{z; \mathbf{x}^\circ}(u_1, \dots, u_n) \in \mathbb{Z}[u_1, \dots, u_n]$ not divisible by any variable u_i . The vector $\mathbf{g}_{z; \mathbf{x}^\circ}$ (resp. the polynomial $F_{z; \mathbf{x}^\circ}$) is called the *g-vector* (resp the *F-polynomial*) of z with respect to \mathbf{x}° .

The *F-polynomials* are conjectured (and in many cases proved) to have positive coefficients. However, even with this conjecture not yet proved in complete generality, one can still evaluate every $F_{z; \mathbf{x}^\circ}$ at an n -tuple of elements of an arbitrary semifield \mathbb{P} , since, as shown in [10], $F_{z; \mathbf{x}^\circ}(u_1, \dots, u_n)$ can be expressed as a subtraction-free rational expression in u_1, \dots, u_n . We denote such an evaluation as $F_{z; \mathbf{x}^\circ}|_{\mathbb{P}}$. Using this notation, one can use *g-vectors* and *F-polynomials* to compute all the cluster variables in an *arbitrary* cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ with the coefficients in an

arbitrary semifield \mathbb{P} . Namely, consider again the cluster algebra $\mathcal{A}^\circ = \mathcal{A}(\mathbf{x}^\circ, \mathbf{y}^\circ, B)$ with principal coefficients at the initial seed with the same exchange matrix B . Then, as shown in [10, Corollary 6.3], every cluster variable $z^\circ \in \mathcal{A}^\circ$ gives rise to a cluster variable $z \in \mathcal{A}$ given by

$$(4.7) \quad z = x_1^{g_1} \cdots x_n^{g_n} \frac{F_{z^\circ; \mathbf{x}^\circ}(\widehat{y}_1, \dots, \widehat{y}_n)}{F_{z^\circ; \mathbf{x}^\circ}|_{\mathbb{P}}(y_1, \dots, y_n)},$$

and all the cluster variables in \mathcal{A} are of this form; here the elements \widehat{y}_j of the ambient field for \mathcal{A} have the same meaning as in (4.5). Comparing (4.7) and (4.6), we obtain the following assertion.

Proposition 4.5. *In the above notation, suppose that the elements $y_1, \dots, y_n \in \mathbb{P}$ are multiplicatively independent, i.e., the correspondence $y_j^\circ \mapsto y_j$ identifies \mathbb{P}° with a multiplicative subgroup of \mathbb{P} . Then the cluster algebra \mathcal{A}° can be identified with a $\mathbb{Z}\mathbb{P}^\circ$ -subalgebra of \mathcal{A} via the correspondence sending each cluster variable $z^\circ \in \mathcal{A}^\circ$ to $F_{z^\circ; \mathbf{x}^\circ}|_{\mathbb{P}}(y_1, \dots, y_n)z \in \mathcal{A}$. With this identification, \mathcal{A} is obtained from \mathcal{A}° by the extension of scalars from $\mathbb{Z}\mathbb{P}^\circ$ to $\mathbb{Z}\mathbb{P}$.*

Definition 4.6. A cluster algebra $\mathcal{A} = \mathcal{A}(\mathcal{S})$ is of *finite type* if the mutation equivalence class \mathcal{S} consists of finitely many seeds.

A classification of cluster algebras of finite type was given in [9]. We present it in the form convenient for our current purposes.

Theorem 4.7 ([9]). *A cluster algebra \mathcal{A} is of finite type if and only if the exchange matrix at some seed of \mathcal{A} is of the form $B(c)$ (see (1.4)) for some Coxeter element in the Weyl group associated to a Cartan matrix A of finite type. Furthermore, the type of A in the Cartan-Killing nomenclature is uniquely determined by the mutation equivalence class of seeds in \mathcal{A} , and, for a given A , all the matrices $B(c)$ associated to different Coxeter elements are mutation equivalent to each other.*

By Theorem 4.7, the property of a cluster algebra \mathcal{A} to be of finite type does not depend on the choice of a coefficient system. As shown in [9], the same is true for the structure of the set \mathcal{X} of cluster variables and its division into clusters. To be more precise, fix a Cartan matrix A of the same type as \mathcal{A} , and choose an initial seed $(\mathbf{x}, \mathbf{y}, B)$ with the exchange matrix $B = B(t)$, where $t = t_+ t_-$ is the bipartite Coxeter element (see (2.9)). As before, let Φ be the root system associated with A . Then the cluster variables in \mathcal{A} are in a natural bijection $\alpha \mapsto x[\alpha]$ with the set $\Phi_{\geq -1}$ of “almost positive” roots (the union of the set of positive roots and the set of negative simple roots $-\alpha_i$). (Recall that, for each $\alpha \in \Phi_{\geq -1}$, the integer vector in \mathbb{Z}^I with the components $d_i = [\alpha : \alpha_i]$ is the *denominator vector* in the Laurent expansion of the cluster variable $x[\alpha]$ with respect to the initial cluster.) In particular, each initial cluster variable takes the form $x_i = x[-\alpha_i]$.

To describe the clusters in \mathcal{A} , we recall the two involutive permutations τ_+, τ_- of $\Phi_{\geq -1}$ given by

$$(4.8) \quad \tau_\varepsilon(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_j \text{ with } \varepsilon(j) = -\varepsilon; \\ t_\varepsilon(\alpha) & \text{otherwise.} \end{cases}$$

As shown in [8, Section 3.1], for any $\alpha, \beta \in \Phi_{\geq -1}$, there is a well-defined nonnegative integer $(\alpha \| \beta)$ (*compatibility degree*) uniquely characterized by the following two properties:

$$(4.9) \quad (-\alpha_i \| \beta) = [[\beta : \alpha_i]]_+,$$

$$(4.10) \quad (\tau_\varepsilon \alpha \| \tau_\varepsilon \beta) = (\alpha \| \beta),$$

for any $\alpha, \beta \in \Phi_{\geq -1}$, any $i \in I$, and any sign ε . We say that α and β are *compatible* if $(\alpha \| \beta) = 0$ (this is in fact a symmetric relation). With this terminology in place, the following was shown in [9].

Proposition 4.8. *Under the above parameterization of cluster variables by almost positive roots, the clusters in \mathcal{A} are exactly the families $\mathbf{x}(C) = \{x[\alpha] : \alpha \in C\}$, where C runs over all maximal by inclusion subsets of $\Phi_{\geq -1}$ consisting of mutually compatible roots.*

Note that every cluster $\mathbf{x}(C)$ is included in a unique seed $(\mathbf{x}(C), \mathbf{y}(C), B(C))$ in \mathcal{A} . To emphasize the dependence on \mathcal{A} , we use the notation $(\mathbf{x}(C), \mathbf{y}(C), B(C)) = (\mathbf{x}(C)^{(\mathcal{A})}, \mathbf{y}(C)^{(\mathcal{A})}, B(C))$; note that the exchange matrix $B(C)$ is independent of \mathcal{A} , and it is explicitly described in [9, Definition 4.2].

Now we introduce a tool to relate to each other two cluster algebras of the same finite type but with different coefficient systems. This is a version of the *coefficient specialization* discussed in [10, Section 12]. We will deal only with cluster algebras of geometric type. Let us start with a simple general proposition.

Proposition 4.9. *Let $\mathbb{P} = \text{Trop}(u_j : j \in J)$ and $\tilde{\mathbb{P}} = \text{Trop}(u_{\tilde{j}} : \tilde{j} \in \tilde{J})$ be tropical semifields, and $\varphi : \tilde{\mathbb{P}} \rightarrow \mathbb{P}$ a homomorphism of multiplicative groups. Then φ is a homomorphism of semifields if and only if each $\varphi(u_{\tilde{j}}) \in \mathbb{P}$ is a monomial with nonnegative exponents in the generators u_j of \mathbb{P} , and no two such monomials have a common factor u_j for some $j \in J$.*

Proof. This follows at once from the equalities $u_{\tilde{j}} \oplus 1 = 1$ and $u_{\tilde{j}} \oplus u_{\tilde{j}'} = 1$ for $\tilde{j} \neq \tilde{j}'$. \square

Now let $\tilde{\mathcal{A}}$ be a cluster algebra of finite type as above, with the coefficient semifield $\tilde{\mathbb{P}} = \text{Trop}(u_{\tilde{j}} : \tilde{j} \in \tilde{J})$. In the above notation, the seeds of $\tilde{\mathcal{A}}$ are of the form $(\mathbf{x}(C)^{(\tilde{\mathcal{A}})}, \mathbf{y}(C)^{(\tilde{\mathcal{A}})}, B(C))$. Let $\varphi : \tilde{\mathbb{P}} \rightarrow \mathbb{P}$ be a homomorphism of tropical semifields as in Proposition 4.9. Using φ , we can form a cluster algebra \mathcal{A} of the same type as $\tilde{\mathcal{A}}$, but with coefficients in \mathbb{P} : its seeds are of the form $(\mathbf{x}(C)^{(\mathcal{A})}, \mathbf{y}(C)^{(\mathcal{A})}, B(C))$, where the coefficient tuple $\mathbf{y}(C)^{(\mathcal{A})}$ at some (equivalently, any) seed is taken as $\varphi(\mathbf{y}(C)^{(\tilde{\mathcal{A}})})$. The following relationship between \mathcal{A} and $\tilde{\mathcal{A}}$ is immediate from the definitions.

Proposition 4.10. *The semifield homomorphism φ induces an isomorphism of $\mathbb{Z}\mathbb{P}$ -algebras*

$$\varphi_\star : \tilde{\mathcal{A}} \otimes_{\mathbb{Z}\tilde{\mathbb{P}}} \mathbb{Z}\mathbb{P} \rightarrow \mathcal{A}$$

given by $\varphi_\star(x[\alpha]^{(\tilde{\mathcal{A}})} \otimes 1) = x[\alpha]^{(\mathcal{A})}$ for $\alpha \in \Phi_{\geq -1}$.

Turning to Theorem 1.2, we will show that Proposition 4.10 applies to $\tilde{\mathcal{A}}_{\mathbb{C}} = \mathbb{C}[G^{c,c^{-1}}]$ and $\mathcal{A}_{\mathbb{C}} = \mathbb{C}[L^{c,c^{-1}}]$, where $\tilde{\mathcal{A}}_{\mathbb{C}}$ and $\mathcal{A}_{\mathbb{C}}$ are obtained from $\tilde{\mathcal{A}}$ and \mathcal{A} by

extension of scalars from \mathbb{Z} to \mathbb{C} . First of all, the fact that the coordinate ring of $G^{c,c^{-1}}$ carries the structure of a (complexified) geometric type cluster algebra $\tilde{\mathcal{A}}_{\mathbb{C}}$ of the same finite type as G , was established in [2, Example 2.24]. We will use a slightly modified description of its initial seed by choosing a different reduced word for (c, c^{-1}) . Namely, we have:

- the ambient field for $\tilde{\mathcal{A}}_{\mathbb{C}}$ is just the field of rational functions on $G^{c,c^{-1}}$;
- the coefficient semifield $\tilde{\mathbb{P}}$ is the tropical semifield with the $2n$ generators $\Delta_{c\omega_i, \omega_i}|_{G^{c,c^{-1}}}$ and $\Delta_{\omega_i, c\omega_i}|_{G^{c,c^{-1}}}$ for $i \in I$;
- the initial cluster consists of the functions $\Delta_{\omega_i, \omega_i}|_{G^{c,c^{-1}}}$;
- the initial coefficient tuple consists of the restrictions to $G^{c,c^{-1}}$ of the functions $\Delta_{\omega_j, c\omega_j} \Delta_{c\omega_j, \omega_j} \prod_{i \prec_c j} (\Delta_{\omega_i, c\omega_i} \Delta_{c\omega_i, \omega_i})^{a_{i,j}}$;
- the initial exchange matrix is $B(c)$.

Now recall that by Proposition 3.3, the reduced double cell $L^{c,c^{-1}}$ is obtained from $G^{c,c^{-1}}$ by specializing each invertible regular function $\Delta_{c\omega_i, \omega_i}$ to 1. Algebraically, this means that the coordinate ring $\mathbb{C}[L^{c,c^{-1}}]$ can be described as follows. Let \mathbb{P} be the tropical semifield with the n generators $\Delta_{\omega_i, c\omega_i}|_{L^{c,c^{-1}}}$ for $i \in I$, and let $\varphi : \tilde{\mathbb{P}} \rightarrow \mathbb{P}$ be the restriction homomorphism from $G^{c,c^{-1}}$ to $L^{c,c^{-1}}$. Then φ is as in Proposition 4.9, acting on the generators by

$$\varphi(\Delta_{c\omega_i, \omega_i}|_{G^{c,c^{-1}}}) = 1, \quad \varphi(\Delta_{\omega_i, c\omega_i}|_{G^{c,c^{-1}}}) = \Delta_{\omega_i, c\omega_i}|_{L^{c,c^{-1}}};$$

and we have

$$\mathbb{C}[L^{c,c^{-1}}] = \mathbb{C}[G^{c,c^{-1}}] \otimes_{\tilde{\mathbb{P}}} \mathbb{Z}\mathbb{P}.$$

Comparing this with Proposition 4.10, we conclude that $\mathbb{C}[L^{c,c^{-1}}]$ is the complexified cluster algebra $\mathcal{A}_{\mathbb{C}}$ with the coefficient semifield \mathbb{P} and the initial seed as in Theorem 1.2.

This is still not quite what we need to prove Theorem 1.2, since the auxiliary addition in \mathbb{P} is *different* from the one in $\mathbb{P}^\circ = \text{Trop}(y_{j;c} : j \in I)$. However, the elements $y_{j;c} \in \mathbb{P}$ are related with the generators $\Delta_{\omega_i, c\omega_i}|_{L^{c,c^{-1}}}$ by a triangular (hence invertible) monomial transformation. Therefore, as a multiplicative group, \mathbb{P}° coincides with \mathbb{P} . By Proposition 4.5, the cluster algebra \mathcal{A}° with principal coefficients at the initial seed can be identified with \mathcal{A} (with the cluster variables in \mathcal{A}° obtained by rescaling from those in \mathcal{A}). This concludes the proof of Theorem 1.2.

5. PROOFS OF THEOREMS 1.4 AND 1.5

In this section we fix an indecomposable Cartan matrix A of finite type, and freely use the root system formalism developed in Section 2, and the cluster algebra formalism developed in Section 4 (as before, lifting the restriction that A is indecomposable presents no problem).

Let us recap a little. For a Coxeter element c in the Weyl group of A , let $\mathcal{A}^\circ(c)$ be the cluster algebra with principal coefficients at an initial seed $(\mathbf{x}^\circ, \mathbf{y}^\circ, B(c))$ as defined in Definition 4.4. We have already shown in Theorem 1.2 that the complexification of $\mathcal{A}^\circ(c)$ can be identified with the coordinate ring $\mathbb{C}[L^{c,c^{-1}}]$, and under this identification, the initial cluster variables $x_i^\circ = x_{\omega_i;c}$ and the initial coefficients $y_j^\circ = y_{j;c}$ are given by (1.5) and (1.6), respectively. We have also proved that the

elements $x_{\gamma;c}$ (for $\gamma \in \Pi(c)$) and $y_{j;c}$ satisfy algebraic relations (1.10) and (1.11) in $\mathcal{A}^\circ(c)$. Thus, to prove Theorem 1.4 and to complete the proof of Theorem 1.5, we can forget about the geometric realization of $\mathcal{A}^\circ(c)$, and it remains to prove the following two statements:

(5.1) The elements $x_{\gamma;c} \in \mathcal{A}^\circ(c)$ (for $\gamma \in \Pi(c)$) satisfying (1.10) and (1.11) are exactly the cluster variables in $\mathcal{A}^\circ(c)$.

(5.2) The relations (1.10) and (1.11) are exactly the primitive exchange relations in $\mathcal{A}^\circ(c)$.

For (5.2) recall from [10, Section 12] that an exchange relation (4.4) is called *primitive* if one of the products of cluster variables in the right hand side is empty, understood to be equal to 1.

We start with some combinatorial preparation. First we transfer the compatibility degree function given by (4.9) and (4.10) from $\Phi_{\geq -1}$ to each $\Pi(c)$. Let τ_c be the permutation of $\Pi(c)$ given by

$$(5.3) \quad \tau_c(c^{m-1}\omega_i) = c^m\omega_i \quad (1 \leq m \leq h(i; c)), \quad \tau_c(-\omega_i) = \omega_i.$$

Proposition 5.1. *There is a unique assignment of a nonnegative integer $(\gamma\|\delta)_c$ to any pair of weights $\gamma, \delta \in \Pi(c)$, satisfying the following properties:*

- (1) $(\omega_i\|\omega_j)_c = 0 \quad (i, j \in I)$,
- (2) $(\omega_i\|\gamma)_c = [c^{-1}\gamma - \gamma : \alpha_i] \quad (\gamma \in \Pi(c) - \{\omega_j : j \in I\})$,
- (3) $(\tau_c\gamma\|\tau_c\delta)_c = (\gamma\|\delta)_c \quad (\gamma, \delta \in \Pi(c))$.

We call $(\gamma\|\delta)_c$ the *c-compatibility degree* of γ and δ .

Proof. The uniqueness of the *c-compatibility degree* with desired properties is clear. Note also that, in view of (2.5), condition (2) in Proposition 5.1 can be rewritten as

$$(5.4) \quad (\omega_i\|c^m\omega_j)_c = [c^{m-1}\beta_j : \alpha_i] \quad (i, j \in I; 1 \leq m \leq h(j; c)),$$

where the root β_j is given by (2.4).

In view of (2.8), to prove the existence it suffices to do it for the bipartite Coxeter element $t = t_+t_-$, and then to show that the existence of the *c-compatibility degree* with desired properties implies that of \tilde{c} -compatibility degree, where \tilde{c} is obtained from c as in (2.7).

We start by dealing with $\Pi(t)$. Consider the permutation $\tau = \tau_+\tau_-$ of $\Phi_{\geq -1}$, where τ_+ and τ_- are given by (4.8).

Lemma 5.2. *There is a bijection $\psi : \Pi(t) \rightarrow \Phi_{\geq -1}$ uniquely determined by the properties that $\psi(\omega_i) = -\alpha_i$ for all $i \in I$, and $\psi \circ \tau_t = \tau \circ \psi$. Explicitly, ψ is given by*

$$(5.5) \quad \psi(\gamma) = \begin{cases} -\alpha_i & \text{if } \gamma = \omega_i \text{ for some } i \in I; \\ t^{-1}\gamma - \gamma & \text{otherwise.} \end{cases}$$

Proof. The uniqueness of ψ with the desired properties is clear. The fact that (5.5) defines a bijection between $\Pi(t)$ and $\Phi_{\geq -1}$ follows from Lemmas 2.1 and 2.2. It remains to prove that the map ψ given by (5.5) satisfies the property that $\psi(\tau_t\gamma) =$

$\tau\psi(\gamma)$ for $\gamma \in \Pi(t)$. This is clear if both $\psi(\gamma)$ and $\psi(\tau_t\gamma)$ fall into the second case in (5.5), i.e., if γ is not of the form $\pm\omega_i$. By the definitions, we also have

$$\psi(\tau_t\omega_i) = \tau\psi(\omega_i) = \begin{cases} \alpha_i & \text{if } \varepsilon(i) = +1; \\ t_+\alpha_i & \text{if } \varepsilon(i) = -1, \end{cases}$$

and $\psi(\tau_t(-\omega_i)) = \tau\psi(-\omega_i) = -\alpha_i$ for all $i \in I$, finishing the proof. \square

By Lemma 5.2, one can define the t -compatibility degree by setting

$$(5.6) \quad (\gamma\|\delta)_t = (\psi(\gamma)\|\psi(\delta)) \quad (\gamma, \delta \in \Pi(t)),$$

and it satisfies the desired properties.

Now suppose that \tilde{c} is obtained from c via (2.7). Without loss of generality, we assume that $I = \{1, \dots, n\}$, $c = s_1 s_2 \cdots s_n$, and $\tilde{c} = s_2 \cdots s_n s_1$. It remains to show that the existence of the c -compatibility degree with the desired properties implies that of the \tilde{c} -compatibility degree. The following lemma is an easy consequence of (2.15).

Lemma 5.3. *There is a bijection $\psi_{c,\tilde{c}} : \Pi(\tilde{c}) \rightarrow \Pi(c)$ uniquely determined by the properties that*

$$(5.7) \quad \psi_{c,\tilde{c}}(\omega_i) = \begin{cases} \omega_i & \text{if } i \neq 1; \\ c\omega_i & \text{if } i = 1, \end{cases}$$

and $\psi_{c,\tilde{c}} \circ \tau_{\tilde{c}} = \tau_c \circ \psi_{c,\tilde{c}}$. Explicitly, for $\gamma \in \Pi(\tilde{c})$ we have

$$(5.8) \quad \psi_{c,\tilde{c}}(\gamma) = \begin{cases} \omega_1 & \text{if } \gamma = -\omega_1; \\ s_1\gamma & \text{otherwise.} \end{cases}$$

Now we define the \tilde{c} -compatibility degree by setting

$$(5.9) \quad (\gamma\|\delta)_{\tilde{c}} = (\psi_{c,\tilde{c}}(\gamma)\|\psi_{c,\tilde{c}}(\delta))_c \quad (\gamma, \delta \in \Pi(\tilde{c})).$$

This makes condition (3) in Proposition 5.1 obvious. It remains to check conditions (1) and (2).

The equality $(\omega_i\|\omega_j)_{\tilde{c}} = 0$ is obvious if $i, j \neq 1$. Let us show that $(\omega_1\|\omega_i)_{\tilde{c}} = 0$ and $(\omega_i\|\omega_1)_{\tilde{c}} = 0$ for $i \neq 1$. Indeed, we have

$$\begin{aligned} (\omega_1\|\omega_i)_{\tilde{c}} &= (c\omega_1\|\omega_i)_c = (\omega_1\| -\omega_i)_c \\ &= [\omega_i - c^{-1}\omega_i : \alpha_1] = [s_n s_{n-1} \cdots s_{i+1} \alpha_i : \alpha_1] = 0, \end{aligned}$$

and

$$(\omega_i\|\omega_1)_{\tilde{c}} = (\omega_i\|c\omega_1)_c = [\omega_i - c\omega_1 : \alpha_i] = [\alpha_1 : \alpha_i] = 0.$$

It remains to prove that $(\omega_i\|\tilde{c}^m\omega_j)_{\tilde{c}}$ for $i, j \in \{1, \dots, n\}$ and $1 \leq m \leq h(j; \tilde{c})$ is given by (5.4), i.e., we have

$$(5.10) \quad (\omega_i\|\tilde{c}^m\omega_j)_{\tilde{c}} = [\tilde{c}^{m-1}s_2 \cdots s_{j-1}\alpha_j : \alpha_i],$$

with the convention that for $j = 1$, the index $j - 1$ is understood to be equal to n .

In checking (5.10), we will repeatedly use the following obvious property: the coefficient $[\alpha : \alpha_i]$ does not change under replacing α with $s_k \alpha$ for $k \neq i$. Now let us consider four separate cases.

Case 1. Let $i, j \neq 1$. Then we have

$$\begin{aligned} (\omega_i \| \tilde{c}^m \omega_j)_{\tilde{c}} &= (\omega_i \| c^m \omega_j)_c = [c^{m-1} s_1 \cdots s_{j-1} \alpha_j : \alpha_i] \\ &= [s_1 \tilde{c}^{m-1} s_2 \cdots s_{j-1} \alpha_j : \alpha_i] = [\tilde{c}^{m-1} s_2 \cdots s_{j-1} \alpha_j : \alpha_i], \end{aligned}$$

proving (5.10).

Case 2. Let $i = 1$ and $j \neq 1$. If $m \geq 2$ then we have

$$\begin{aligned} (\omega_1 \| \tilde{c}^m \omega_j)_{\tilde{c}} &= (\omega_1 \| c^{m-1} \omega_j)_c = [c^{m-2} s_1 \cdots s_{j-1} \alpha_j : \alpha_1] \\ &= [s_2 \cdots s_n c^{m-2} s_1 \cdots s_{j-1} \alpha_j : \alpha_1] = [\tilde{c}^{m-1} s_2 \cdots s_{j-1} \alpha_j : \alpha_1], \end{aligned}$$

proving (5.10). And if $m = 1$ then

$$(\omega_1 \| \tilde{c} \omega_j)_{\tilde{c}} = (\omega_1 \| \omega_j)_c = 0 = [s_2 \cdots s_{j-1} \alpha_j : \alpha_1],$$

again proving (5.10).

Case 3. Let $i \neq 1$ and $j = 1$. Then we have

$$\begin{aligned} (\omega_i \| \tilde{c}^m \omega_1)_{\tilde{c}} &= (\omega_i \| c^{m+1} \omega_1)_c = [c^m \alpha_1 : \alpha_i] \\ &= [s_1 \tilde{c}^{m-1} s_2 \cdots s_n \alpha_1 : \alpha_i] = [\tilde{c}^{m-1} s_2 \cdots s_n \alpha_1 : \alpha_i], \end{aligned}$$

proving (5.10).

Case 4. Let $i = j = 1$. Then we have

$$\begin{aligned} (\omega_1 \| \tilde{c}^m \omega_1)_{\tilde{c}} &= (\omega_1 \| c^m \omega_1)_c = [c^{m-1} \alpha_1 : \alpha_1] \\ &= [s_2 \cdots s_n c^{m-1} \alpha_1 : \alpha_1] = [\tilde{c}^{m-1} s_2 \cdots s_n \alpha_1 : \alpha_1], \end{aligned}$$

proving (5.10) in this case as well.

This completes the proof of Proposition 5.1. \square

We now show that the c -compatibility degree satisfies “Langlands duality.” Let Φ^\vee denote the dual root system to Φ ; it corresponds to the transpose Cartan matrix A^T . The Weyl group of A^T is identified with the Weyl group W of A , and there is a canonical W -equivariant bijection $\alpha \mapsto \alpha^\vee$ between Φ and Φ^\vee . This bijection restricts to a bijection between $\Phi_{\geq -1}$ and $\Phi_{\geq -1}^\vee$. As shown in [8, Proposition 3.3], we have

$$(5.11) \quad (\alpha \| \beta) = (\beta^\vee \| \alpha^\vee) \quad (\alpha, \beta \in \Phi_{\geq -1}).$$

Now consider the set $\Pi(c)^\vee$ defined in the same way as $\Pi(c)$ but associated with the dual root system. Thus, the elements of $\Pi(c)^\vee$ are coweights of the form $c^m \omega_i^\vee$, where the ω_i^\vee are fundamental coweights, and $0 \leq m \leq h(i; c)$ (it easily follows from Lemmas 2.1 and 2.2 that the numbers $h(i; c)$ for the dual root system Φ^\vee are the same as for Φ). The correspondence $\omega_i \mapsto \omega_i^\vee$ uniquely extends to the τ_c -equivariant bijection $\gamma \mapsto \gamma^\vee$ between $\Pi(c)$ and $\Pi(c)^\vee$. By the definition, for every $\gamma \in \Pi(c) - \{\omega_i : i \in I\}$, we have

$$(5.12) \quad (\tau_c^{-1} \gamma - \gamma)^\vee = \tau_c^{-1} \gamma^\vee - \gamma^\vee.$$

With some abuse of notation, we denote by $(\gamma^\vee \parallel \delta^\vee)_c$ the c -compatibility degree on $\Pi(c)^\vee$.

Proposition 5.4. *For $\gamma, \delta \in \Pi(c)$, we have*

$$(5.13) \quad (\gamma \parallel \delta)_c = (\delta^\vee \parallel \gamma^\vee)_c.$$

Proof. The equality (5.13) follows at once from (5.11) and an obvious fact that each of the bijections ψ and $\psi_{c,\tilde{c}}$ (see Lemmas 5.2 and 5.3) commutes with passing to the dual roots and weights. \square

We illustrate the use of (5.13) by the following lemma to be used later.

Lemma 5.5. *For any index $j \in I$, and $\gamma \in \Pi(c) - \{\omega_j, c\omega_j\}$, we have*

$$(5.14) \quad (\gamma \parallel c\omega_j)_c + \sum_{i \prec_c j} a_{i,j}(\gamma \parallel c\omega_i)_c = -((\gamma \parallel \omega_j)_c + \sum_{j \prec_c i} a_{i,j}(\gamma \parallel \omega_i)_c).$$

Furthermore, for $\gamma = \omega_j$, the right hand side of (5.14) is equal to 0, while the left hand side is equal to 1; and for $\gamma = c\omega_j$, the left hand side of (5.14) is equal to 0, while the right hand side is equal to -1 .

Proof. First of all, if $\gamma = \omega_k$ for some k , then the right hand side of (5.14) is equal to 0 by condition (1) in Proposition 5.1. As for the left hand side, by condition (2) it is equal to

$$[\beta_j + \sum_{i \prec_c j} a_{i,j}\beta_i : \alpha_k] = [\alpha_j : \alpha_k] = \delta_{jk}$$

(see (2.6)). The case $\gamma = c\omega_k$ is treated similarly: now the left hand side of (5.14) is equal to 0, while the right hand side is equal to $-\delta_{jk}$ (by applying (2.6) with c replaced by c^{-1}).

It remains to consider the case when γ is not of the form ω_k or $c\omega_k$. Let $\alpha = c^{-1}\gamma - \gamma$. Using (5.13), we can rewrite the right hand side of (5.14) as

$$-([\alpha^\vee : \alpha_j^\vee] + \sum_{j \prec_c i} a_{i,j}[\alpha^\vee : \alpha_i^\vee]) = -\langle \alpha^\vee, \omega_j + \sum_{j \prec_c i} a_{i,j}\omega_i \rangle,$$

where $\langle ?, ? \rangle$ is the standard W -invariant pairing between the coweights and weights. Similarly, the left hand side of (5.14) can be rewritten as

$$\langle c^{-1}\alpha^\vee, \omega_j + \sum_{i \prec_c j} a_{i,j}\omega_i \rangle = \langle \alpha^\vee, c(\omega_j + \sum_{i \prec_c j} a_{i,j}\omega_i) \rangle.$$

It remains to check the identity

$$c(\omega_j + \sum_{i \prec_c j} a_{i,j}\omega_i) = -(\omega_j + \sum_{j \prec_c i} a_{i,j}\omega_i).$$

Using (2.2), (2.5) and (2.6), we obtain

$$\begin{aligned} -(\omega_j + \sum_{j \prec_c i} a_{i,j}\omega_i) &= -\alpha_j + \omega_j + \sum_{i \prec_c j} a_{i,j}\omega_i \\ &= (\omega_j - \beta_j) + \sum_{i \prec_c j} a_{i,j}(\omega_i - \beta_i) = c(\omega_j + \sum_{i \prec_c j} a_{i,j}\omega_i), \end{aligned}$$

finishing the proof. \square

Returning to the proofs of (5.1) and (5.2), we will deduce these statements from the following proposition of independent interest.

Proposition 5.6. *For every Coxeter element c , there exists a cluster algebra $\tilde{\mathcal{A}}(c)$ of geometric type, satisfying the following properties:*

- (1) *The coefficient semifield of $\tilde{\mathcal{A}}(c)$ is $\tilde{\mathbb{P}}(c) = \text{Trop}(p[\gamma] : \gamma \in \Pi(c))$.*
- (2) *The cluster variables in $\tilde{\mathcal{A}}(c)$ are labeled by the set $\Pi(c)$, the variable corresponding to $\gamma \in \Pi(c)$ being denoted $x[\gamma]$.*
- (3) *The initial seed of $\tilde{\mathcal{A}}(c)$ is of the form $(\mathbf{x}, \mathbf{y}, B(c))$, where $\mathbf{x} = (x[\omega_i] : i \in I)$, and $\mathbf{y} = (y_j : j \in I)$ with*

$$(5.15) \quad y_j = p[\omega_j] p[c\omega_j]^{-1} \prod_{\gamma \in \Pi(c) - \{\omega_j, c\omega_j\}} p[\gamma]^{(\gamma \| c\omega_j)_c + \sum_{i \prec_{c,j}} a_{i,j}(\gamma \| c\omega_i)_c}.$$

- (4) *The primitive exchange relations in $\tilde{\mathcal{A}}(c)$ are exactly the following (for $k \in I$ and $1 \leq m \leq h(k; c)$):*

$$(5.16) \quad x[-\omega_k] x[\omega_k] = p[\omega_k] \prod_{i \prec_{c,k}} x[\omega_i]^{-a_{i,k}} \prod_{k \prec_c i} x[-\omega_i]^{-a_{i,k}} + \prod_{\gamma \in \Pi(c)} p[\gamma]^{(\gamma \| \omega_k)_c};$$

$$(5.17) \quad x[c^{m-1}\omega_k] x[c^m\omega_k] = p[c^m\omega_k] \prod_{i \prec_{c,k}} x[c^m\omega_i]^{-a_{i,k}} \prod_{k \prec_c i} x[c^{m-1}\omega_i]^{-a_{i,k}} \\ + \prod_{\gamma \in \Pi(c)} p[\gamma]^{(\gamma \| c^m\omega_k)_c}.$$

Before proving Proposition 5.6, let us show that it implies the desired statements (5.1) and (5.2). Indeed, let $\mathbb{P}^\circ = \text{Trop}(y_i^\circ : i \in I)$ be the coefficient semifield of the cluster algebra with principal coefficients $\mathcal{A}^\circ(c)$ (see Definition 4.4). Let $\varphi : \tilde{\mathbb{P}}(c) \rightarrow \mathbb{P}^\circ$ be the tropical semifield homomorphism acting on the generators as follows:

$$(5.18) \quad \varphi(p[\gamma]) = \begin{cases} y_i & \text{if } \gamma = \omega_i; \\ 1 & \text{otherwise.} \end{cases}$$

Applying Proposition 4.10 to this homomorphism, we see that the cluster variables in $\mathcal{A}^\circ(c)$ are in a natural bijection with those in $\tilde{\mathcal{A}}(c)$, and so can be labeled by the same set $\Pi(c)$. If we denote by $x_{\gamma;c}$ the cluster variable in $\mathcal{A}^\circ(c)$ corresponding to $x[\gamma] \in \tilde{\mathcal{A}}(c)$, then in view of Proposition 4.10, both (5.1) and (5.2) become consequences of the following statement: under the homomorphism φ , the relations (5.16) and (5.17) specialize respectively to (1.10) and (1.11). But this is immediate from the definition of the c -compatibility degree.

Proof of Proposition 5.6. We obtain the required assertion as a consequence of the following two statements:

- (1) For the bipartite Coxeter element t , the cluster algebra $\tilde{\mathcal{A}}(t)$ with required properties can be identified (by renaming the cluster variables and the generators of the coefficient semifield) with the cluster algebra with *universal coefficients* constructed in [10, Theorem 12.4].

- (2) If $\tilde{\mathcal{A}}(c)$ is the cluster algebra with required properties for $c = s_1 \cdots s_n$, and $\tilde{c} = s_2 \cdots s_n s_1$, then $\tilde{\mathcal{A}}(\tilde{c})$ can be identified (as above) with $\tilde{\mathcal{A}}(c)$.

Thus, each $\tilde{\mathcal{A}}(c)$ for different choices of a Coxeter element c is a realization of the same cluster algebra with universal coefficients, but with a different choice of an acyclic initial seed (with the exchange matrix $B(c)$).

To prove (1), let us first recall the nomenclature of cluster variables in [10]. Following [10, Definition 9.1], for every $i \in I$ and $m \in \mathbb{Z}$ such that $\varepsilon(i) = (-1)^m$, we define a root $\alpha(i; m)$, by setting, for all $r \geq 0$:

$$(5.19) \quad \alpha(i; r) = \underbrace{t_+ t_+ \cdots t_{\varepsilon(i)}}_{r \text{ factors}} (-\alpha_i) \quad \text{for } \varepsilon(i) = (-1)^r;$$

$$(5.20) \quad \alpha(j; -r - 1) = \underbrace{t_+ t_- \cdots t_{\varepsilon(j)}}_{r \text{ factors}} (-\alpha_j) \quad \text{for } \varepsilon(j) = (-1)^{r-1}.$$

In particular, we have $\alpha(i; m) = -\alpha_i$ for $m \in \{0, -1\}$.

The following proposition is a consequence of a classical result of R. Steinberg [15] (cf. [8, Lemma 2.1, Proposition 2.5]).¹

Proposition 5.7. *The roots $\alpha(i; m)$ with $m \in [-h - 1, -2]$, and $\varepsilon(i) = (-1)^m$ are positive, distinct, and every positive root is of such a form. Furthermore, for $m \in [0, h + 1]$, and $\varepsilon(i) = (-1)^m$, we have $\alpha(i; m) = \alpha(i^*; m - h - 2)$.*

In [10], the cluster variables are denoted by $x_{i; m}$ with $i \in I$, $m \in \mathbb{Z}$ and $\varepsilon(i) = (-1)^m$. If we parameterize them by the set $\Phi_{\geq -1}$ as in Section 4 above (that is, using denominator vectors), then by [10, Theorem 10.3], each $x_{i; m}$ for $m \in [-h - 1, h + 1]$ will correspond to the root $\alpha(i; m)$, and so we will write $x_{i; m} = x[\alpha(i; m)]$. Comparing (5.20) with the description of the bijection $\psi : \Pi(t) \rightarrow \Phi_{\geq -1}$ given in Lemma 5.2, it is easy to see that, for $0 \leq m \leq h(i; t)$, we have

$$(5.21) \quad \psi(t^m \omega_i) = \begin{cases} \alpha(i; -2m) & \text{if } \varepsilon(i) = +1; \\ \alpha(i; -2m - 1) & \text{if } \varepsilon(i) = -1. \end{cases}$$

According to [10, (8.12), (10.11)], the primitive exchange relations are those having the product $x_{j; m-1} x_{j; m+1}$ in the left hand side. Renaming each cluster variable $x_{i; m}$ in these relations as $x[\gamma]$, where $\gamma \in \Pi(t)$ is such that $\psi(\gamma) = \alpha(i; m)$, and ignoring the coefficients, an easy check using (5.21) shows that the cluster variables appear in these relations in exactly the same way as in (5.16) and (5.17).

To deal with coefficients, recall that the generators of the coefficient semifield in [10, Theorem 12.4] are of the form $p[\alpha^\vee]$ with $\alpha^\vee \in \Phi_{\geq -1}^\vee$. We identify $\Phi_{\geq -1}^\vee$ with $\Pi(t)$ by means of the following modification of Lemma 5.2:

Lemma 5.8. *There is a bijection $\psi^\vee : \Pi(t) \rightarrow \Phi_{\geq -1}^\vee$ uniquely determined by the properties that $\psi^\vee(\omega_i) = \varepsilon(i) \alpha_i^\vee$ for all $i \in I$, and $\psi^\vee \circ \tau_t = \tau^{-1} \circ \psi^\vee$. Explicitly, ψ^\vee is given by*

$$(5.22) \quad \psi^\vee(\gamma) = (\tau_+ \psi(\gamma))^\vee,$$

where ψ is as in Lemma 5.2.

¹Unfortunately, the corresponding result in [10, Proposition 9.3] was stated incorrectly.

Proof. The uniqueness statement is clear, as well as the fact that (5.22) defines a bijection $\Pi(t) \rightarrow \Phi_{\geq -1}^\vee$. Using Lemma 5.2 and (4.8), we obtain that this bijection satisfies

$$\psi^\vee(\omega_i) = (\tau_+(-\alpha_i))^\vee = \varepsilon(i)\alpha_i^\vee,$$

and

$$\psi^\vee \tau_t(\gamma) = (\tau_+ \tau \psi(\gamma))^\vee = (\tau_- \psi(\gamma))^\vee = (\tau^{-1} \tau_+ \psi(\gamma))^\vee = \tau^{-1} \psi^\vee(\gamma),$$

as desired. \square

As a consequence of (5.6), (4.10) and (5.22), we have

$$(5.23) \quad (\gamma \| \delta)_t = (\psi^\vee(\delta) \| \psi^\vee(\gamma)) \quad (\gamma, \delta \in \Pi(t)),$$

Now everything is ready to check the following: replacing $p[\gamma]$ with $p[\psi^\vee(\gamma)]$ for all $\gamma \in \Pi(t)$ transforms each element y_j given by (5.15) (with $c = t$) into the element

$$y_{j;0} = \prod_{\alpha^\vee \in \Phi_{\geq -1}^\vee} p[\alpha^\vee]^{\varepsilon(j)[\alpha^\vee : \alpha_j^\vee]}$$

in [10, (12.5)].

Case 1. Let $\varepsilon(j) = +1$. Then we have $\psi^\vee(\omega_j) = \alpha_j^\vee$ and $\psi^\vee(t\omega_j) = -\alpha_j^\vee$. Since there are no indices i with $i \prec_t j$, if $\psi^\vee(\gamma) = \alpha^\vee \neq \pm\alpha_j^\vee$, then the exponent of $p[\gamma]$ in the right hand side of (5.15) is equal to

$$(\gamma \| t\omega_j)_t = (-\alpha_j^\vee \| \alpha^\vee) = [\alpha^\vee : \alpha_j^\vee].$$

Thus, the replacement of $p[\gamma]$ with $p[\alpha^\vee]$ indeed transforms y_j into $y_{j;0}$.

Case 2. Let $\varepsilon(j) = -1$. Then we have $\psi^\vee(\omega_j) = -\alpha_j^\vee$ and $\psi^\vee(t\omega_j) = \alpha_j^\vee$. In this case, there are no indices i with $j \prec_t i$, so by Lemma 5.5, if $\psi^\vee(\gamma) = \alpha^\vee \neq \pm\alpha_j^\vee$, then the exponent of $p[\gamma]$ in the right hand side of (5.15) is equal to

$$-(\gamma \| \omega_j)_t = -(-\alpha_j^\vee \| \alpha^\vee) = -[\alpha^\vee : \alpha_j^\vee],$$

proving our claim in this case as well.

To finish the proof of (1), it remains to check that the same replacement of each $p[\gamma]$ with $p[\psi^\vee(\gamma)]$ transforms the coefficients in the relations (5.16) and (5.17) into the coefficients of the corresponding relations in the cluster algebra from [10, Theorem 12.4]. Pick some $\delta \in \Pi(t)$, and let $\alpha^\vee = \psi^\vee(\delta)$. Recall from [10, Section 12] that $p[\alpha^\vee]$ is the primitive coefficient (the one at the non-trivial product of cluster variables) in the relation between $x[\tau_- \alpha]$ and $x[\tau_+ \alpha]$. This agrees with the fact that $p[\delta]$ is the primitive coefficient in the relation (of the form (5.16) or (5.17)) between $x[\tau_t^{-1} \delta]$ and $x[\delta]$. It remains to observe that the constant term in the same relation (5.16) or (5.17), that is, the element $\prod_{\gamma \in \Pi(t)} p[\gamma]^{(\gamma \| \delta)_t}$, transforms, in view of (5.23), into $\prod_{\beta^\vee \in \Phi_{\geq -1}^\vee} p[\beta^\vee]^{(\alpha^\vee \| \beta^\vee)}$, which is the constant term in the corresponding relation in [10] (see [10, (12.18)], where unfortunately there is an annoying typo: the correct exponent on the right is $(\alpha^\vee \| \beta^\vee)$ instead of $(\beta^\vee \| \alpha^\vee)$). This completes the check of property (1) above.

To prove (2), it is enough to show the following: after replacing, for each $\gamma \in \Pi(\tilde{c})$, the element $x[\gamma] \in \tilde{\mathcal{A}}(\tilde{c})$ by $x[\psi_{c,\tilde{c}}(\gamma)] \in \tilde{\mathcal{A}}(c)$, and the coefficient $p[\gamma] \in \tilde{\mathbb{P}}(\tilde{c})$ by

$p[\psi_{c,\tilde{c}}(\gamma)] \in \tilde{\mathbb{P}}(c)$ (see Lemma 5.3), the properties (3) and (4) for $\tilde{\mathcal{A}}(\tilde{c})$ in Proposition 5.6 become identical to the corresponding properties for $\tilde{\mathcal{A}}(c)$.

Starting with property (4), it is enough to check it for the relations (5.16), since the relations (5.17) are obtained from them by repeatedly applying the transformation $\tau_{\tilde{c}}$ (resp. τ_c) to all occurring labeling weights $\gamma \in \Pi(\tilde{c})$ (resp. $\gamma \in \Pi(c)$), and the bijection $\psi_{c,\tilde{c}} : \Pi(\tilde{c}) \rightarrow \Pi(c)$ intertwines $\tau_{\tilde{c}}$ with τ_c . The check that every relation (5.16) for $\tilde{\mathcal{A}}(\tilde{c})$ turns into one of the (5.16) or (5.17) for $\tilde{\mathcal{A}}(c)$ after applying $\psi_{c,\tilde{c}}$ is immediate from the definitions.

To deal with the remaining property (3), we note that $\psi_{c,\tilde{c}}$ sends the initial cluster $(x[\omega_1], \dots, x[\omega_n])$ in $\tilde{\mathcal{A}}(\tilde{c})$ into the cluster $(x[c\omega_1], x[\omega_2], \dots, x[\omega_n])$ in $\tilde{\mathcal{A}}(c)$ obtained from the initial one by the mutation μ_1 . Remembering (1.4) and (4.2), we see that μ_1 sends the exchange matrix $B(c)$ into $B(\tilde{c})$. It remains to show that the transformation (4.3) (for $k = 1$) applied to the coefficient tuple in $\tilde{\mathbb{P}}(c)$ given by (5.15) produces the elements y'_j obtained from the corresponding $\tilde{y}_j \in \tilde{\mathbb{P}}(\tilde{c})$ by applying $\psi_{c,\tilde{c}}$ to all the labeling weights.

For $j = 1$, we can express y_1 as

$$y_1 = p[c\omega_1]^{-1} \prod_{\gamma \in \Pi(c)} p[\gamma]^{(\gamma \| c\omega_1)_c};$$

therefore, we have

$$y'_1 = y_1^{-1} = p[c\omega_1] \prod_{\gamma \in \Pi(c)} p[\gamma]^{-(\gamma \| c\omega_1)_c}.$$

On the other hand, using (5.14), we can express $\tilde{y}_1 \in \tilde{\mathbb{P}}(\tilde{c})$ as

$$\tilde{y}_1 = p[\omega_1] \prod_{\gamma \in \Pi(\tilde{c})} p[\gamma]^{-(\gamma \| \omega_1)_{\tilde{c}}},$$

which indeed transforms into y'_1 by applying $\psi_{c,\tilde{c}}$ to all its labeling weights.

Finally, for $j > 1$, we have $b_{1,j} = -a_{1,j} \geq 0$, hence (4.3) results in

$$\begin{aligned} y'_j &= y_j y_1^{-a_{1,j}} p[c\omega_1]^{-a_{1,j}} = y_j \prod_{\gamma \in \Pi(c)} p[\gamma]^{-a_{1,j}(\gamma \| c\omega_1)_c} \\ &= p[\omega_j] p[c\omega_j]^{-1} \prod_{\gamma \in \Pi(c) - \{\omega_j, c\omega_j\}} p[\gamma]^{(\gamma \| c\omega_j)_c + \sum_{1 \neq i <_{c,j} a_{i,j}(\gamma \| c\omega_i)_c}, \end{aligned}$$

which is again obtained from \tilde{y}_j by applying $\psi_{c,\tilde{c}}$ to all its labeling weights.

This finishes the proof of Proposition 5.6, hence also the proofs of Theorems 1.4 and 1.5. \square

In the rest of the section we give some corollaries of the developed techniques. First, since the c -compatibility degree for every Coxeter element c is obtained by a chain of bijections from the compatibility degree in [8, Section 3.1] (see (5.6) and (5.9)), combining the above results with Proposition 4.8 yields the following corollary.

Corollary 5.9. *For an arbitrary Coxeter element c and any $\gamma, \delta \in \Pi(c)$, the cluster variables $x_{\gamma;c}$ and $x_{\delta;c}$ in $\mathcal{A}(c)$ belong to the same cluster if and only if $(\gamma \| \delta)_c = 0$ (in particular, the latter condition is symmetric in γ and δ).*

Second, combining the arguments in the proof of Proposition 5.6 with Proposition 4.5, we conclude that all the algebras with principal coefficients $\mathcal{A}(c)$ for different choices of the Coxeter element c are isomorphic to each other. To be more specific, let us again suppose that $c = s_1 s_2 \cdots s_n$ and $\tilde{c} = s_2 \cdots s_n s_1$. Then the isomorphism $\tilde{\mathcal{A}}(\tilde{c}) \rightarrow \tilde{\mathcal{A}}(c)$ constructed above gives rise to the isomorphism $\mathcal{A}(\tilde{c}) \rightarrow \mathcal{A}(c)$ given as follows.

Corollary 5.10. *The correspondence in Proposition 4.5 establishes an isomorphism of cluster algebras $\varphi : \mathcal{A}(\tilde{c}) \rightarrow \mathcal{A}(c)$ acting on the initial cluster variables $x_{\omega_i; \tilde{c}}$ and the initial coefficient tuple $(y_{1; \tilde{c}}, \dots, \tilde{y}_{n; \tilde{c}})$ in $\mathcal{A}(\tilde{c})$ as follows:*

$$\begin{aligned} \varphi(x_{\omega_i; \tilde{c}}) &= x_{\omega_i; c}, & \varphi(y_{i; \tilde{c}}) &= y_{i; c} y_{1; c}^{-a_{1, i}} \quad (i \neq 1), \\ \varphi(x_{\omega_1; \tilde{c}}) &= x_{c\omega_1; c} = x_{\omega_1; c}^{-1} (y_{1; c} + \prod_{i \neq 1} x_{\omega_i; c}^{-a_{i, 1}}), & \varphi(y_{1; \tilde{c}}) &= y_{1; c}^{-1}. \end{aligned}$$

Remark 5.11. Since, in view of Theorem 1.2, the algebra $\mathcal{A}(c)$ is the coordinate ring of the variety $L^{c, c^{-1}}$, the isomorphism $\varphi : \mathcal{A}(\tilde{c}) \rightarrow \mathcal{A}(c)$ gives rise to a biregular isomorphism $\varphi^* : L^{c, c^{-1}} \rightarrow L^{\tilde{c}, \tilde{c}^{-1}}$. This isomorphism can be described as follows. Note that the definition (3.12) and the standard properties of double Bruhat cells imply the following statement: if in the Weyl group W we have factorizations $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_k$ such that $\ell(u) = \ell(u_1) + \cdots + \ell(u_k)$ and $\ell(v) = \ell(v_1) + \cdots + \ell(v_k)$, then the product map in G induces an open embedding

$$L^{u_1, v_1} \times \cdots \times L^{u_k, v_k} \hookrightarrow L^{u, v}.$$

In particular, denoting $c_o = s_2 \cdots s_n$, we have open embeddings

$$L^{s_1, e} \times L^{c_o, c_o^{-1}} \times L^{e, s_1} \hookrightarrow L^{c, c^{-1}}, \quad L^{e, s_1} \times L^{c_o, c_o^{-1}} \times L^{s_1, e} \hookrightarrow L^{\tilde{c}, \tilde{c}^{-1}}.$$

Note also that

$$L^{s_1, e} = \{x_{-1}(u) : u \in \mathbb{C}^*\}, \quad L^{e, s_1} = \{x_1(t) : t \in \mathbb{C}^*\}.$$

Now we claim that φ^* restricts to an isomorphism

$$L^{s_1, e} \cdot L^{c_o, c_o^{-1}} \cdot L^{e, s_1} \rightarrow L^{e, s_1} \cdot L^{c_o, c_o^{-1}} \cdot L^{s_1, e}$$

given by

$$(5.24) \quad \varphi^*(x_{-1}(u)x_o x_1(t)) = x_1(u)x_o x_{-1}(t^{-1}) \quad (u, t \in \mathbb{C}^*, x_o \in L^{c_o, c_o^{-1}}).$$

It suffices to prove (5.24) for x_o of the form $x_{-2}(u_2) \cdots x_{-n}(u_n)x_n(t_n) \cdots x_2(t_2)$ with all u_i and t_i nonzero complex numbers, since such elements form an open dense subset of $L^{c_o, c_o^{-1}}$. Then (5.24) can be checked by a straightforward calculation using (3.25), (3.26) and the commutation relations in [3, Proposition 7.2]; we leave the details to the reader.

6. PROOFS OF THEOREMS 1.8, 1.10 AND 1.12, AND THEIR COROLLARIES

In this section we work with the cluster algebra $\mathcal{A}(c)$ from Theorem 1.2. When it is convenient, we assume without further warning that the index set I is $[1, n]$, and the Coxeter element c under consideration is $s_1 \cdots s_n$.

Proof of Theorem 1.8. For $\gamma \in \Pi(c)$, let $\mathbf{d}(\gamma)$ denote the denominator vector of the cluster variable $x_{\gamma;c}$ with respect to the initial cluster $(x_{\omega_i;c} : i \in I)$. We identify \mathbb{Z}^n with the root lattice Q using the basis of simple roots, and so assume that $\mathbf{d}(\gamma) \in Q$. In view of [10, (7.6), (7.7)], the vectors $\mathbf{d}(\gamma)$ are uniquely determined from the initial conditions

$$(6.1) \quad \mathbf{d}(\omega_i) = -\alpha_i,$$

and the recurrence relations

$$(6.2) \quad \mathbf{d}(c^m \omega_j) + \mathbf{d}(c^{m-1} \omega_j) = [-\sum_{i \prec_c j} a_{i,j} \mathbf{d}(c^m \omega_i) - \sum_{j \prec_c i} a_{i,j} \mathbf{d}(c^{m-1} \omega_i)]_+$$

(for all $j \in I$ and $1 \leq m \leq h(j; c)$), which follow from (1.11); here the notation $[v]_+$ for $v \in Q$ is understood component-wise, i.e.,

$$[\sum a_i \alpha_i]_+ = \sum [a_i]_+ \alpha_i.$$

We need to show that the solution of the relations (6.2) with the initial conditions (6.1) is given by

$$(6.3) \quad \mathbf{d}(c^m \omega_j) = c^{m-1} \omega_j - c^m \omega_j = c^{m-1} \beta_j \quad (j \in I, 1 \leq m \leq h(j; c))$$

(see Lemma 2.1).

First let us check that the values given by (6.1) and (6.3) satisfy (6.2) for $m = 1$. Indeed, the right hand side of (6.2) is equal to

$$[-\sum_{i \prec_c j} a_{i,j} \beta_i + \sum_{j \prec_c i} a_{i,j} \alpha_i]_+ = -\sum_{i \prec_c j} a_{i,j} \beta_i,$$

since each $\beta_i = s_1 \cdots s_{i-1} \alpha_i$ is a positive linear combination of the roots $\alpha_{i'}$ with $i' < i$ (in any total order on the index set I compatible with the relation $i' \prec_c i$). On the other hand, the left hand side of (6.2) is equal to $\beta_j - \alpha_j$. So the two sides are equal to each other by (2.6).

For $m \geq 2$, the desired identity (6.2) takes the form

$$c^{m-1} \beta_j + c^{m-2} \beta_j = -\sum_{i \prec_c j} a_{i,j} c^{m-1} \beta_i - \sum_{j \prec_c i} a_{i,j} c^{m-2} \beta_i,$$

which can be simplified to

$$(6.4) \quad \beta_j + \sum_{i \prec_c j} a_{i,j} \beta_i = -c^{-1}(\beta_j + \sum_{j \prec_c i} a_{i,j} \beta_i).$$

Now the left hand side of (6.4) is equal to α_j by (2.6). On the other hand, since

$$-c^{-1} \beta_i = -c^{-1} s_1 \cdots s_{i-1} \alpha_i = -s_n \cdots s_{i+1} s_i \alpha_i = s_n \cdots s_{i+1} \alpha_i,$$

the right hand side of (6.4) is also equal to α_j by the same equality (2.6) with the Coxeter element c replaced by c^{-1} . This completes the proof of Theorem 1.8. \square

Proof of Corollary 1.9. The first assertion is clear, and the second one follows at once by combining the expression for denominator vectors given in Theorem 1.8 with Corollary 5.9 and formula (2) in Proposition 5.1. \square

Proof of Theorem 1.10. Recall that the \mathbf{g} -vector $\mathbf{g}_{z;\mathbf{x}} = (g_1, \dots, g_n) \in \mathbb{Z}^n$ and the F -polynomial $F_{z;\mathbf{x}}(t_1, \dots, t_n) \in \mathbb{Z}[t_1, \dots, t_n]$ of a cluster variable $z \in \mathcal{A}^\circ$ with respect to the initial cluster \mathbf{x} are defined by (4.6). As prescribed by Theorem 1.10, we now identify \mathbb{Z}^n with the weight lattice P using the basis of fundamental weights. Following [10], we introduce the P -(multi)grading in the Laurent polynomial ring $\mathbb{Z}[\mathbf{x}^{\pm 1}, \mathbf{y}^{\pm 1}]$ by setting

$$(6.5) \quad \deg(x_{\omega_j;c}) = \omega_j, \quad \deg(y_{j;c}) = - \sum_{i \in I} b_{i,j} \omega_i = \sum_{i \prec_c j} a_{i,j} \omega_i - \sum_{j \prec_c i} a_{i,j} \omega_i.$$

Then by (4.5), all elements \widehat{y}_j° are homogeneous of degree 0, hence the same is true for the factor $F_{z;\mathbf{x}^\circ}(\widehat{y}_1^\circ, \dots, \widehat{y}_n^\circ)$ in (4.6), so the \mathbf{g} -vector is equal to $\mathbf{g}_{z;\mathbf{x}} = \deg(z)$.

Following (3.15), let us write a generic element of $L^{c,c^{-1}}$ in the form

$$x_{-1}(u_1) \cdots x_{-n}(u_n) x_n(t_n) \cdots x_1(t_1),$$

and view all t_j and u_j as rational functions on $L^{c,c^{-1}}$. In view of (3.25) and (3.26), the functions t_j and u_j are Laurent monomials in the initial cluster variables $x_{\omega_i;c}$ and the coefficients $y_{i;c}$, and we have

$$(6.6) \quad \deg(u_j) = -\deg(x_{\omega_j;c}) = -\omega_j, \\ \deg(t_j) = \deg(y_{j;c}) + \deg(u_j) + \sum_{i \prec_c j} a_{i,j} \deg(u_i) = -(\omega_j + \sum_{j \prec_c i} a_{i,j} \omega_i).$$

Remembering (1.5), Theorem 1.10 can be restated as follows: with respect to the P -grading given by (6.6), we have

$$(6.7) \quad \deg(\Delta_{\gamma,\gamma}(x_{-1}(u_1) \cdots x_{-n}(u_n) x_n(t_n) \cdots x_1(t_1))) = \gamma.$$

To prove (6.7), we write each factor $x_{-j}(u_j)$ as $x_{\bar{j}}(u_j) u_j^{-\alpha_j^\vee}$ (see (3.14)), and move all the factors $u_j^{-\alpha_j^\vee}$ all the way to the right, using the commutation relations in (3.24). We obtain

$$x_{-1}(u_1) \cdots x_{-n}(u_n) x_n(t_n) \cdots x_1(t_1) = x_{\bar{1}}(w_1) \cdots x_{\bar{n}}(w_n) x_n(v_n) \cdots x_1(v_1) \prod_{i \in I} u_i^{-\alpha_i^\vee},$$

where the w_j and v_j are given by

$$(6.8) \quad w_j = u_j \prod_{i \prec_c j} u_i^{a_{i,j}}, \quad v_j = t_j \prod_{i=1}^n u_i^{-a_{i,j}}.$$

Expressing γ in the form $\gamma = \sum_i g_i \omega_i$ and using (3.3), we obtain

$$\begin{aligned} & \Delta_{\gamma,\gamma}(x_{-1}(u_1) \cdots x_{-n}(u_n) x_n(t_n) \cdots x_1(t_1)) \\ &= \Delta_{\gamma,\gamma}(x_{\bar{1}}(w_1) \cdots x_{\bar{n}}(w_n) x_n(v_n) \cdots x_1(v_1)) \cdot \left(\prod_i u_i^{-\alpha_i^\vee} \right)^\gamma \\ &= \Delta_{\gamma,\gamma}(x_{\bar{1}}(w_1) \cdots x_{\bar{n}}(w_n) x_n(v_n) \cdots x_1(v_1)) \prod_i u_i^{-g_i}, \end{aligned}$$

hence

$$\begin{aligned} & \deg(\Delta_{\gamma,\gamma}(x_{-1}(u_1) \cdots x_{-n}(u_n) x_n(t_n) \cdots x_1(t_1))) \\ &= \gamma + \deg(\Delta_{\gamma,\gamma}(x_{\bar{1}}(w_1) \cdots x_{\bar{n}}(w_n) x_n(v_n) \cdots x_1(v_1))). \end{aligned}$$

Thus it remains to prove that

$$\deg(\Delta_{\gamma,\gamma}(x_{\bar{1}}(w_1) \cdots x_{\bar{n}}(w_n)x_n(v_n) \cdots x_1(v_1))) = 0 .$$

Using (6.8) and (6.6), we deduce that

$$\deg(w_j) = -\omega_j - \sum_{i \prec_c j} a_{i,j} \omega_i = -\deg(v_j) .$$

Now for every nonzero complex numbers w_1, \dots, w_n there is $a \in H$ such that $w_j = a^{\alpha_j}$ for all j . Using (3.3) and (3.24), we conclude that

$$\begin{aligned} & \Delta_{\gamma,\gamma}(x_{\bar{1}}(w_1) \cdots x_{\bar{n}}(w_n)x_n(v_n) \cdots x_1(v_1)) \\ &= \Delta_{\gamma,\gamma}(ax_{\bar{1}}(w_1) \cdots x_{\bar{n}}(w_n)x_n(v_n) \cdots x_1(v_1)a^{-1}) \\ &= \Delta_{\gamma,\gamma}(x_{\bar{1}}(1) \cdots x_{\bar{n}}(1)x_n(w_nv_n) \cdots x_1(w_1v_1)) . \end{aligned}$$

Since each product $w_j v_j$ has degree 0, this completes the proof of Theorem 1.10. \square

Proof of Theorem 1.12. We use the notation in the proof of Theorem 1.10. In view of (4.6), the F -polynomial $F_{z;\mathbf{x}}(t_1, \dots, t_n)$ is obtained from the expansion of the cluster variable $z = x_{\gamma;c}$ by specializing all initial cluster variables $x_{\omega_i;c}$ to 1, and each coefficient $y_{i;c}$ to t_i . In view of (3.25) and (3.26), this amounts to specializing all u_i to 1 in $\Delta_{\gamma,\gamma}(x_{-1}(u_1) \cdots x_{-n}(u_n)x_n(t_n) \cdots x_1(t_1))$, which is exactly the desired assertion. \square

Proof of Corollary 1.13. The constant term of the polynomial $F_{z;\mathbf{x}}(t_1, \dots, t_n)$ is obtained by specializing all t_i to 0 in (1.15), i.e., is equal to $\Delta_{\gamma,\gamma}(x_{\bar{1}}(1) \cdots x_{\bar{n}}(1))$. The fact that the latter is equal to 1 follows at once by the representation-theoretic arguments in the above proof of (3.20). \square

Remark 6.1. We have already mentioned in Remark 1.11 that the set $\Pi(c)$ is identical with the set of generators of the c -Cambrian fan studied in [14]. In particular, our Theorem 1.10 implies [14, Theorem 10.2], proved there only modulo [10, Conjecture 7.12]. Comparing our Corollary 5.9 with [14, Theorem 10.1], we obtain the following description of the cones in the c -Cambrian fan: they are exactly the cones generated by subsets $\{\gamma_1, \dots, \gamma_k\} \subset \Pi(c)$ such that $(\gamma_i \parallel \gamma_j)_c = 0$ for all $i, j = 1, \dots, k$.

7. PROOF OF THEOREM 1.1

In this section we assume that $G = SL_{n+1}(\mathbb{C})$ is of type A_n , and the Coxeter element c is equal to $s_1 \cdots s_n$ in the standard numbering of simple roots. As usual, the Weyl group W is identified with the symmetric group S_{n+1} , so that s_i becomes a simple transposition $(i, i+1)$. Then c is the cycle $(1, 2, \dots, n+1)$.

In this case, the generalized minors $\Delta_{\gamma,\delta}$ specialize to the ordinary minors (i.e., determinants of square submatrices) as follows. The weights in $W\omega_k$ are in bijection with the k -subsets of $[1, n+1] = \{1, \dots, n+1\}$, so that $W = S_{n+1}$ acts on them in a natural way, and ω_k corresponds to $[1, k]$. If γ and δ correspond to k -subsets I and J , respectively, then $\Delta_{\gamma,\delta} = \Delta_{I,J}$ is the minor with the row set I and the column set J .

Now let L denote the subvariety of G consisting of tridiagonal matrices M of the form (1.1) (with all y_1, \dots, y_n non-zero). We start our proof of Theorem 1.1 by

showing that L is indeed the reduced double cell $L^{c,c^{-1}}$. We use the characterization of $L^{c,c^{-1}}$ given by Propositions 3.2 and 3.3. It is well-known that the Bruhat order on k -elements subsets of $[1, n+1]$ is component-wise: if $I = \{i_1 < \dots < i_k\}$ and $J = \{j_1 < \dots < j_k\}$ then $I \leq J$ means that $i_\nu \leq j_\nu$ for $\nu = 1, \dots, k$. Thus, we can rewrite Propositions 3.2 and 3.3 as follows.

Corollary 7.1. *A matrix $M \in SL_{n+1}(\mathbb{C})$ belongs to the reduced double Bruhat cell $L^{c,c^{-1}}$ for $c = s_1 \dots s_n$ if and only if it satisfies the following conditions:*

- (1) $\Delta_{I,[1,k]} = \Delta_{[1,k],I} = 0$ for $k = 1, \dots, n$, and all subsets $I = \{i_1 < \dots < i_k\} \subset [1, n+1]$ such that $i_\nu > \nu + 1$ for some $\nu = 1, \dots, k$.
- (2) $\Delta_{[1,k],[2,k+1]} \neq 0$, and $\Delta_{[2,k+1],[1,k]} = 1$ for $k = 1, \dots, n$.

To prove that $L = L^{c,c^{-1}}$, we start with the inclusion $L \subseteq L^{c,c^{-1}}$. Let $M = (m_{i,j}) \in L$. Since M is tridiagonal, it satisfies condition (1) in Corollary 7.1 because every term in the expansion of $\Delta_{I,[1,k]}(M)$ or $\Delta_{[1,k],I}(M)$ (with I as in this condition) contains at least one matrix entry $m_{i,j}$ with $|i - j| > 1$. To prove condition (2), note that any tridiagonal matrix M becomes triangular after removing the first column and the last row (or the first row and the last column), implying that

$$(7.1) \quad \begin{aligned} \Delta_{[1,k],[2,k+1]}(M) &= m_{1,2}m_{2,3} \dots m_{k,k+1} \\ \Delta_{[2,k+1],[1,k]}(M) &= m_{2,1}m_{3,2} \dots m_{k+1,k}. \end{aligned}$$

So for $M \in L$, we have $\Delta_{[1,k],[2,k+1]}(M) = y_1 \dots y_k \neq 0$, and $\Delta_{[2,k+1],[1,k]}(M) = 1$, as required.

To prove the reverse inclusion $L^{c,c^{-1}} \subseteq L$, first let us show that every matrix in $L^{c,c^{-1}}$ is tridiagonal. It is enough to check this for a generic element of the form (3.15). Note that in our situation the matrix $x_{-k}(u)$ (resp. $x_k(t)$) is obtained from the identity matrix by replacing the 2×2 submatrix with rows and columns k and $k+1$ by $\begin{pmatrix} u^{-1} & 0 \\ 1 & u \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$). Then one can check that the product

$$x_{-1}(u_1) \dots x_{-n}(u_n) x_n(t_n) \dots x_1(t_1)$$

is tridiagonal by a direct matrix computation (or better yet, by using the graphical formalism for computing determinants of such products developed in [6, Proof of Theorem 12]).

Once we know that every matrix $M \in L^{c,c^{-1}}$ is tridiagonal, the remaining conditions $m_{k,k+1} \neq 0$ and $m_{k+1,k} = 1$ follow at once from (7.1) and condition (2) in Corollary 7.1. This concludes the proof of the equality $L = L^{c,c^{-1}}$.

We continue the proof of Theorem 1.1. Part (1) of Theorem 1.1 is a special case of Theorem 1.2. We need only to observe that the expression for $y_{j;c}$ in (1.6) specializes to the restriction to L of the function $\Delta_{[1,j],[2,j+1]}/\Delta_{[1,j-1],[2,j]}$, which, in view of (7.1) is equal to the above-diagonal matrix entry y_j (see (1.1)).

Part (2) of Theorem 1.1 is a special case of Theorem 1.4. We need only to observe that in our case $h(k;c) = n+1-k$, and, for $0 \leq m \leq n+1-k$, the weight $c^m \omega_k \in \Pi(c)$ corresponds to the subset $[m+1, m+k] \subset [1, n+1]$.

Turning to Part (3), we first note that the primitive exchange relations (1.10) and (1.11) specialize to the following relations, which are among those in (1.2):

$$(7.2) \quad x_{[1,k]}x_{[k+1,n+1]} = y_k x_{[1,k-1]}x_{[k+2,n+1]} + 1;$$

$$(7.3) \quad x_{[m,m+k-1]}x_{[m+1,m+k]} = y_m y_{m+1} \cdots y_{m+k-1} + x_{[m,m+k]}x_{[m+1,m+k-1]},$$

where $k \in [1, n]$, $1 \leq m \leq h(k; c) = n + 1 - k$. To prove Theorem 1.1, it remains to show that the exchange relations in $\mathcal{A}(c)$ are exactly those in (1.2).

We start by recalling the geometric interpretation of the cluster variables and clusters in type A_n given in [9, Section 12.2]. Namely, the cluster variables can be associated with the diagonals of a regular $(n + 3)$ -gon \mathbf{P}_{n+3} , and the clusters are all maximal collections of mutually non-crossing diagonals, which are naturally identified with the *triangulations* of \mathbf{P}_{n+3} . We label the vertices of \mathbf{P}_{n+3} by numbers $1, \dots, n + 3$ in the counter-clockwise order, and denote by $x_{\langle i,j \rangle}$ the cluster variable associated with the diagonal $\langle i, j \rangle$ connecting vertices i and j (with the convention that $x_{\langle i,j \rangle} = 1$ if i and j are two adjacent vertices of \mathbf{P}_{n+3}). According to [9, (12.2)], in this realization the exchange relations have the form

$$(7.4) \quad x_{\langle i,k \rangle} x_{\langle j,\ell \rangle} = p_{ik,j\ell}^+ x_{\langle i,j \rangle} x_{\langle k,\ell \rangle} + p_{ik,j\ell}^- x_{\langle i,\ell \rangle} x_{\langle j,k \rangle},$$

where i, j, k, ℓ are any four vertices of \mathbf{P}_{n+3} taken in counter-clockwise order, and $p_{ik,j\ell}^\pm$ are some elements of the coefficient semifield.

Clearly, the set of labels $\Pi(c)$ is in a bijection with the set of all diagonals of \mathbf{P}_{n+3} via the correspondence

$$(7.5) \quad [i, j] \longleftrightarrow \langle i, j + 2 \rangle \quad (1 \leq i \leq j \leq n + 1).$$

Renaming each cluster variable $x_{[i,j]}$ in Theorem 1.1 by $x_{\langle i,j+2 \rangle}$, we see that, if we ignore the coefficients, then the desired relations (1.2) turn into (7.4). Note that, under this relabeling, the initial cluster gets associated with the triangulation of \mathbf{P}_{n+3} by all the diagonals from the vertex 1, see Figure 1 (ignoring for the moment the segments labeled by y_1, y_2 and y_3).

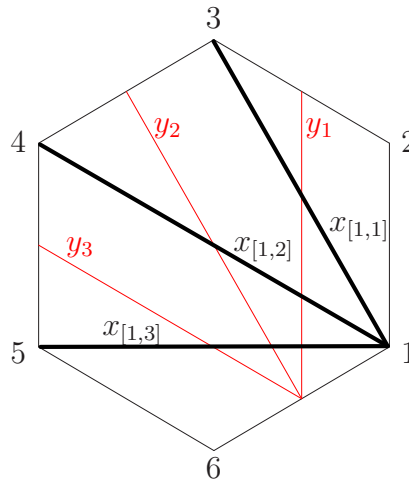


FIGURE 1. Initial seed in type A_3 .

To complete the proof, it remains to verify the coefficients in the relations (1.2). To do this, we use the geometric realization of the universal coefficients (cf. Proposition 5.6 and [10, Theorem 12.4]) due to S. Fomin and A. Zelevinsky (the proof will appear elsewhere).

Consider the dual $(n+3)$ -gon \mathbf{P}'_{n+3} whose vertices are the midpoints of the sides of \mathbf{P}_{n+3} . For $i = 2, \dots, n+3$, we denote by i' the midpoint of the side with vertices $i-1$ and i ; we also denote by $1'$ the midpoint of the side connecting 1 and $n+3$. We refer to the diagonals of \mathbf{P}'_{n+3} as *dual diagonals*.

Proposition 7.2 (S. Fomin, A. Zelevinsky). *After relabeling the cluster variables and the generators of the coefficient semifield in the cluster algebra $\tilde{\mathcal{A}}(c)$ in Proposition 5.6 by the diagonals of \mathbf{P}_{n+3} via the correspondence (7.5), the coefficient $p_{ik,j\ell}^+$ (resp., $p_{ik,j\ell}^-$) in an exchange relation (7.4) turns into the product of the generators $p\langle a, b \rangle$ such that the dual diagonal $\langle a', b' \rangle$ is contained in the strip formed by $\langle i, j \rangle$ and $\langle k, \ell \rangle$ (resp., by $\langle i, \ell \rangle$ and $\langle j, k \rangle$).*

Example 7.3. In type A_3 , the exchange relation between $x\langle 2, 5 \rangle$ and $x\langle 4, 6 \rangle$ has the form

$$x\langle 2, 5 \rangle x\langle 4, 6 \rangle = p\langle 1, 5 \rangle p\langle 2, 5 \rangle x\langle 2, 4 \rangle x\langle 5, 6 \rangle + p\langle 3, 6 \rangle p\langle 4, 6 \rangle x\langle 2, 6 \rangle x\langle 4, 5 \rangle,$$

as illustrated by Figure 2 (where we show only those dual diagonals that contribute to the exchange relation); note that by our convention, $x\langle 4, 5 \rangle = x\langle 5, 6 \rangle = 1$.

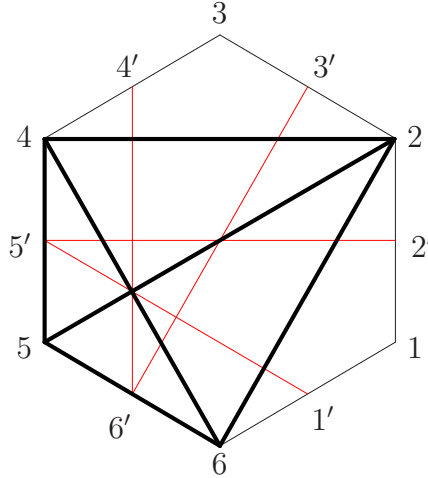


FIGURE 2. Exchange relation between $x\langle 2, 5 \rangle$ and $x\langle 4, 6 \rangle$.

As shown in Section 5, the cluster algebra with principal coefficients $\mathcal{A}(c)$ is obtained from $\tilde{\mathcal{A}}(c)$ by the coefficient specialization (5.18). Using the relabeling in Proposition 7.2, this specialization is described as follows: it sends $p\langle 1, j+2 \rangle$ to y_j for $j = 1, \dots, n$, and the rest of the generators $p\langle i, j \rangle$ are sent to 1. (This specialization is illustrated by Figure 1 showing the dual diagonals associated with y_1, y_2, y_3 .)

Now consider any relation (1.2). Rewriting it in the form (7.4), we obtain

$$x_{\langle i, k+2 \rangle} x_{\langle j, \ell+2 \rangle} = p^+ x_{\langle i, j \rangle} x_{\langle k+2, \ell+2 \rangle} + p^- x_{\langle i, \ell+2 \rangle} x_{\langle j, k+2 \rangle},$$

where the coefficients p^+ and p^- are obtained from those given in Proposition 7.2 by the above specialization. By inspection of the corresponding dual diagonals, we conclude that $p_+ = y_{j-1}y_j \cdots y_k$ and $p_- = 1$, verifying the coefficients in (1.2) and finishing the proof of Theorem 1.1.

We conclude the paper by an example illustrating the correspondence between various reduced double cells given by (5.24).

Example 7.4. Let $c = s_1s_2$ and $\tilde{c} = s_2s_1$ be the two Coxeter elements in type A_2 , and let $L = L^{c,c^{-1}}$ and $\tilde{L} = L^{\tilde{c},\tilde{c}^{-1}}$ be the corresponding reduced double Bruhat cells in $G = SL_3(\mathbb{C})$. The variety L has been already described: it consists of matrices

$$M = \begin{pmatrix} v_1 & y_1 & 0 \\ 1 & v_2 & y_2 \\ 0 & 1 & v_3 \end{pmatrix}$$

with nonzero y_1, y_2, y_3 . Using Propositions 3.2 and 3.3, the variety \tilde{L} can be described as the set of matrices $\tilde{M} = (m_{i,j}) \in G$ satisfying the conditions

- $\Delta_{[1,2],[2,3]}(\tilde{M}) = \Delta_{[2,3],[1,2]}(\tilde{M}) = 0$;
- $m_{1,3} \neq 0$, $\Delta_{[1,2],\{1,3\}}(\tilde{M}) \neq 0$;
- $m_{3,1} = \Delta_{\{1,3\},[1,2]}(\tilde{M}) = 1$.

The correspondence $L \rightarrow \tilde{L}$ in (5.24) sends a generic element

$$M = x_{-1}(u_1)x_{-2}(u_2)x_2(t_2)x_1(t_1) \in L$$

to

$$\tilde{M} = x_1(u_1)x_{-2}(u_2)x_2(t_2)x_{-1}(t_1^{-1}) \in \tilde{L}.$$

Performing the matrix multiplication, we see that M and \tilde{M} are given by

$$M = \begin{pmatrix} u_1^{-1} & u_1^{-1}t_1 & 0 \\ 1 & u_1u_2^{-1} + t_1 & u_1u_2^{-1}t_2 \\ 0 & 1 & u_2 + t_2 \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} u_1u_2^{-1} + t_1 & u_1u_2^{-1}t_1^{-1} & u_1u_2^{-1}t_2 \\ u_2^{-1} & u_2^{-1}t_1^{-1} & u_2^{-1}t_2 \\ 1 & t_1^{-1} & u_2 + t_2 \end{pmatrix}.$$

An easy calculation shows that the correspondence $M \mapsto \tilde{M}$ extends to an isomorphism $L \rightarrow \tilde{L}$ given by

$$M = \begin{pmatrix} v_1 & y_1 & 0 \\ 1 & v_2 & y_2 \\ 0 & 1 & v_3 \end{pmatrix} \mapsto \tilde{M} = \begin{pmatrix} v_2 & v_1v_2y_1^{-1} - 1 & y_2 \\ v_1v_2 - y_1 & v_1(v_1v_2y_1^{-1} - 1) & v_1y_2 \\ 1 & v_1y_1^{-1} & v_3 \end{pmatrix},$$

while the inverse isomorphism $\tilde{L} \rightarrow L$ is given by

$$\begin{aligned} \tilde{M} &= \begin{pmatrix} m_{1,1} & m_{1,1}m_{3,2} - 1 & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ 1 & m_{3,2} & m_{3,3} \end{pmatrix} \mapsto \\ M &= \begin{pmatrix} m_{2,3}m_{1,3}^{-1} & m_{1,1}m_{2,3}m_{1,3}^{-1} - m_{2,1} & 0 \\ 1 & m_{1,1} & m_{1,3} \\ 0 & 1 & m_{3,3} \end{pmatrix}. \end{aligned}$$

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DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MA 02115
E-mail address: yang.s@neu.edu

DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MA 02115
E-mail address: andrei@neu.edu